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Partial differential equations with infinitely many variables

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INTRODUCTION

This thesis is mainly concerned with second order elliptic and parabolic equations with infinitely many variables of the following type:

$$\lambda\psi(x) - \frac{1}{2}\mathrm{Tr}[Q(x)D^2\psi(x)] - < D\psi(x), Ax >= f(x),\ x \in H,\ \lambda > 0, \tag{0.0.1}$$

$$\begin{cases} \partial_t u(t,x) = \dfrac{1}{2}\mathrm{Tr}\ [Q(x)D_x^2 u(t,x)] + < D_x u(t,x), Ax > +F(t,x), & t \in]0,T],\ x \in D(A), \\ u(0,x) = f(x), & x \in H, \end{cases} \tag{0.0.2}$$

where H is a real separable Hilbert space and $f : H \mapsto \mathbb{R}$ belongs to $\mathcal{C}_b(H)$, the space of all real bounded uniformly continuous functions defined on H. Moreover $Q(x)$, $x \in H$, are suitable self-adjoint non negative bounded linear operators on H and A is a linear operator on H, with domain $D(A)$, that generates a $\mathcal{C}_0$-semigroup. We denote by Tr $(Q(x)D^2\psi(x))$, the trace of $Q(x)D^2\psi(x)$, $x \in H$.

Let us notice that if H is finite dimensional, $H = \mathbb{R}^n$, then equations (0.1.1) and (0.1.2) can be written respectively as

$$\lambda\psi(x) - \frac{1}{2}\sum_{i,j=1}^{n} q_{ij}(x)D_{ij}^2\psi(x) - \sum_{i,j=1}^{n} a_{ij}D_i\psi(x)x_j = f(x),\ x \in \mathbb{R}^n,\ \lambda > 0, \tag{0.0.3}$$

$$\begin{cases} \partial_t u(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} q_{ij}(x)D_{ij}^2 u(t,x) + \sum_{i,j=1}^{n} a_{ij}D_i u(t,x)x_j + F(t,x),\ t \in]0,T], \\ u(0,x) = f(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{0.0.4}$$

There is an increasing interest in studying PDE's with infinitely many variables like (0.1.1) and (0.1.2). These equations have applications in Statistical Physics and Field Theory (see the monograph Berezansky and Kondratiev [6] and Stroock [72]). Operators of energy of the simplest physical systems with infinitely many degrees of freedom (e.g., a free boson field or a collection of noninteracting quantum oscillators) are given by second order elliptic differential operators, acting in spaces of functions with infinitely many variables (see Chapters 6 and 7 of Berezansky and Kondratiev [6]).

Moreover in Vishik and Fursikov [84], it has been studied the Hopf equation in infinite dimensions in connection with the Navier-Stokes equations.

Another important motivation to study equations (0.1.1) and (0.1.2) comes from a well known connection with stochastic differential equations as

$$\begin{cases} dX(t) \,=\, AX(t)dt \,+\, Q^{1/2}(X(t))\, dW(t), \quad t \geq 0, \\ X(0) = \, x, \qquad x \in H, \end{cases} \tag{0.0.5}$$

where W is an $H-$valued cylindrical Wiener process on some probability space $(\Omega, \mathcal{F}, P)$. Important physical phenomena, concerning for instance Field Theory and Stochastic Quantization Theory, can be described by means of equations like (0.1.5). We refer to Jona-Lasinio and Mitter [46], [47], Borkar et al [10], Da Prato and Tubaro [22].

Assume that one can solve equation (0.1.5), and denote by $X(\cdot, x)(\omega)$, the corresponding solution. Then setting

$$u(t,x) \,=\, \int_\Omega f(X(t,x)(\omega))\, P(d\omega), \tag{0.0.6}$$

where $f \in \mathcal{C}_b(H)$, it turns out that $u(t,x) = S_t f(x)$ is formally the solution of equation (0.1.2) with $F = 0$. The semigroup S_t is said to be the *Markov transition semigroup*, see Definition 6.2.15, corresponding to (0.1.2). Equation (0.1.2) is called a *Kolmogorov equation* . Moreover the existence of a Markov transition semigroup S_t for (0.1.2) allows to find the solution of (0.1.1), by Laplace transform.

In order to solve (0.1.2) two approaches have been developed: one is deterministic and uses classical methods on PDE's; the other is stochastic and consists of solving first equation (0.1.5) and then using the representation formula (0.1.6), in order to obtain a candidate solution for (0.1.2). The stochastic approach allows to find a solution for degenerate equations as well, however it requires stronger regularity assumptions on the coefficients $Q(x)$.

It is worth noticing that finding a regular solution for (0.1.1) or (0.1.2), by analytic methods, yields uniqueness in law for equation (0.1.5), by a generalization of an argument of Stroock and Varadhan [73] (see Zambotti [86] and [87]).

Assuming that $A = 0$ and $Q(x) = Q$, $x \in H$, where Q is a positive self-adjoint trace-class operator on H, equations (0.1.1) and (0.1.2) were studied in Gross [40], [41] and Dalecky [17], [18]. In Gross [41], there is also an extension of classical Potential Theory to infinite dimensional Dirichlet problems, by using probabilistic arguments and introducing the notion of abstract Wiener space. Later in Piech [62], [63] and in Vishik [83], using the abstract Wiener spaces setting, has been considered the case of

$$Q(x) \,=\, Q^{1/2}(I + G(x))Q^{1/2}, \quad x \in H, \tag{0.0.7}$$

where $G(x)$ is a family of trace-class operators, satisfying strong smoothness assumptions. When $Q(x) = Q$ is a positive self-adjoint trace-class operator on H and A generates a $\mathcal{C}_0$-semigroup on H, equation (0.1.2) was first studied by Dalecky, see Dalecky and Fomin [19], by probabilistic arguments. He has proved the existence and uniqueness of a generalized, non-smooth solution u. Smoothing properties for u have been investigated in Cannarsa and Da Prato [11], using analytic tools and in Da Prato and Zabczyk [23] and [24] by solving the corresponding stochastic equation (0.1.5).

Cannarsa and Da Prato (see [12] and [13]) have studied equation (0.1.1), when G is Hölder-continuous from H with values in the space of all trace class operators. They have also obtained optimal regularity results of Schauder type for the solution.

Recently Kolmogorov equations in infinite dimensions have been investigated in Zabczyk [85]. Moreover in Flandoli and Gozzi [35], perturbing equation (0.1.2), it has been studied the Kolmogorov equation associated with a stochastic Navier-Stokes equation.

We also mention that another possible approach to solve (0.1.1), (0.1.2) and more general equations in $L^p(H, \mu)$, where μ is a suitable Borel measure on H, can be developed by means of Dirichlet forms (we refer to Ma and Rockner [56]).

Finally we remark that existence and uniqueness results for (0.1.2) can be used to construct regular solutions for second order Hamilton-Jacobi equations, arising in stochastic control theory, see for instance Gozzi [38], Gozzi and Rouy [39]. This method can be viewed as an alternative to the viscosity solutions approach developed in P. L. Lions [54] and Swiech [74].

In this thesis we present several original results, contained in the papers Priola [65], [66], [67], [68], [69], Priola and Zambotti [70] along with some results available in the literature.

We study equations (0.1.1) and (0.1.2) in the Banach space $\mathcal{C}_b(H)$ of all real uniformly continuous and bounded functions on H, endowed with the sup norm, where the heat semigroup is strongly continuous. To this aim, we only use analytic tools, mainly Semigroup Theory, some results from Interpolation Theory and basic properties of Gaussian measures in infinite dimensions. We assume that the reader is familiar with basic results from the theory of $\mathcal{C}_0$-semigroups.

The thesis is divided into three parts. The first one is devoted to prove preliminary results, concerning density results in $\mathcal{C}_b(H)$ and some properties of the heat semigroup in infinite dimensions, also needed later. In the second part we consider elliptic equations like (0.1.1) and also a homogeneous infinite dimensional Dirichlet problem in a half space of H. Finally in the third part we study parabolic equations like (0.1.2). To this purpose we introduce the class of π-semigroups. We think that this part can be also read independently of the other parts (with the exception of Chapter 1). We stress that Chapters 5, 6, 7 are the heart of this thesis. Now we briefly discuss the content of each chapter.

In Chapter 1, we review some known results on Gaussian measures in infinite dimensions and basic concepts from Interpolation Theory. We also show, following Priola [67], the equivalence between the spaces $\mathcal{C}^n_Q(H)$, $n \geq 1$, which are a slight modification of those recently introduced by Cannarsa and Da Prato (see [11] and [12]), and the spaces $\mathcal{C}^n_{H_0}(H)$ related to the differentiabilty along a subspace, introduced by Gross in the abstract Wiener spaces setting.

In Chapter 2 we present new density results in $\mathcal{C}_b(H)$, see Priola [65]. They will be frequently used in Parts I and II. In particular in Theorem 2.2.7 we extend a theorem of Lasry and Lions [52]. Note that if H is infinite dimensional then $\mathcal{C}^2_b(H)$ is not dense in $\mathcal{C}_b(H)$ (see Nemirowski and Semenov [59]).

In Chapter 3 we deal with the properties of the heat semigroup O_t, that is the

Markov transition semigroup associated with equation (0.1.2), with $F = A = 0$ and $Q(x) = Q$, $x \in H$, where Q is a trace class operator on H. Our main result provides a new characterization of the domain of the infinitesimal generator of O_t, see Theorem 3.3.2, proved in Priola [67] and [68]. This way we extend a classical result of Gross (see Corollary 3.2 in Gross [41]).

In Chapter 4 we consider equation (0.1.1), with $A = 0$ and the variable coefficients $Q(x)$ given by (0.1.7). Following Priola and Zambotti [70], in Theorems 4.2.2 and 4.3.6 we improve the Schauder estimates of Cannarsa and Da Prato [12]. We show that the solution ψ of (0.1.1) has the second Q-derivative of Hilbert-Schmidt type at any $x \in H$. Our new approach allows to treat (0.1.1), when F is Q-Hölder-continuous from H with values in the space of Hilbert-Schmidt operators, instead of trace class operators.

In Chapter 5 we study equation (0.1.1) in $\mathcal{C}_b(H_+)$, where H_+ is a half space of H. More precisely we study a homogeneous Dirichlet problem on H_+, by considering a Markov transition semigroup P_t, naturally associated with the Dirichlet problem. We consider $A = 0$ and $Q(x) = Q$, $x \in H$, and provide Schauder estimates for the solution. This chapter follows Priola [66], with some improvements.

In Chapter 6, following Priola [67] and [68], we deal with a new class of Markov type semigroups on $\mathcal{C}_b(\Omega)$, where Ω is an open set of H. Transition semigroups corresponding to (0.1.1) and (0.1.2), which are not strongly continuous in general, belong to this class. We call these semigroups, $\pi-$semigroups. We can define a generator $\mathcal{A}$ for a $\pi-$semigroup S_t and show that the resolvent operator of $\mathcal{A}$ is given by the pointwise Laplace transform of S_t, see Proposition 6.2.11. Moreover we show that for any Markov transition $\mathcal{C}_0$-semigroup U_t, the generator of U_t as $\mathcal{C}_0$-semigroup and as π-semigroup coincide (see Corollary 6.2.14).

In Section 6.3 we prove a Hille-Yosida type theorem for this class of semigroups.

In Chapter 7, following Priola [67] and [69], we study the Cauchy problem for $\pi-$semigroups that is related to equation (0.1.2). We define a suitable notion of strict and strong solution and prove existence, uniqueness and regularity theorems. This way we prove in particular uniqueness for the mild solution of (0.1.2), when $Q(x) = Q$, $x \in H$, see Theorem 7.2.5. We also extend the results in Gozzi and Cerrai [15] concerning a problem of approximation for mild solutions of (0.1.2), see Section 7.3 and Section 7.4. These results allow to costruct regular solutions for second order Hamilton-Jacobi equations, see Gozzi [38] and Gozzi and Rouy [39].

Part I

Foundations

Chapter 1

Preliminaries

In this chapter we review well known facts on Gaussian measures and differentiability in Banach spaces along with basic concepts from Interpolation Theory.

We also introduce the main functions spaces that will be used. On this subject in Proposition 1.3.2, following Priola [67], we show the equivalence between the functions spaces $\mathcal{C}^n_Q(H)$, $n \geq 1$, which are a slight modification of those recently introduced in Cannarsa and Da Prato [12], [13], and the spaces $\mathcal{C}^n_{H_0}(H)$, considered in the theory of abstract Wiener spaces (see for instance Elson [32], Gross [40], [41], Kuo [50], Lee [53], Piech [64]).

1.1 Borel measures in infinite dimensions

We now present some general definitions from Measure Theory in Banach spaces (for more details we refer to Parthasarathy [61], Vakhania et al [81]).

Let E be a separable metric space. A finite, positive and countably additive function, defined on the σ-algebra $\mathcal{B}(E)$ of all Borel subsets of E, is called a *Borel measure.*

A Borel measure μ on E is said to be *degenerate* if there exists a non empty open subset Ω of E, such that $\mu(\Omega) = 0$.

Let μ and ν be two Borel measures on E. We say that μ and ν are *singular* if there exists $B \in \mathcal{B}(E)$ such that $\mu(B) = \nu(E \backslash B) = 0$. We say that μ and ν are *equivalent* if μ is absolutely continuous with respect ν and ν is absolutely continuous with respect to μ.

Let (μ_j) be a sequence of Borel measures on E. We say that (μ_j) *converges weakly* to a Borel measure μ, if for any real uniformly continuous and bounded function f on E, we have

$$\lim_{j \to \infty} \int_E f(y)\, \mu_j(dy) \;=\; \int_E f(y)\, \mu(dy). \tag{1.1.1}$$

Let F be another separable metric space and let μ, ν be two Borel measures on E and F respectively. We denote by $\mu \otimes \nu$ the Borel *product measure,* between μ and ν, on the separable metric space $E \times F$. It is uniquely defined by setting

$$\mu \otimes \nu(A \times B) \stackrel{\text{def}}{=} \mu(A)\nu(B), \quad A \in \mathcal{B}(E),\ B \in \mathcal{B}(F).$$

Let now $(A, \mathcal{A}, P)$ be a probability space (i.e. A is a measurable space endowed with the σ-algebra $\mathcal{A}$ and P is a measure on $\mathcal{A}$ such that $P(A) = 1$) and $(X, \|\cdot\|_X)$ be a real Banach space. A map $f : (A, \mathcal{A}, P) \to (X, \mathcal{B}(X))$ is said to be *measurable* if for any $B \in \mathcal{B}(X)$, one has $f^{-1}(B) \in \mathcal{A}$. Moreover f is called *simple* if it is measurable and in addition $f(A)$ is finite.

Let $g : (A, \mathcal{A}, P) \to (X, \mathcal{B}(X))$ be a measurable mapping such that $g(A)$ is separable in X (it follows that g can be approximated pointwise by a sequence of simple functions, see for instance §I.1.4 of Vakhania et al [81]). We say that g is *Bochner integrable* if

$$\int_A \|g(x)\|_X \, P(dx) \; < \; \infty. \tag{1.1.2}$$

A measurable mapping $\xi : A \to \mathbb{R}$ ($\mathbb{R}$ will be always endowed with the σ-algebra $\mathcal{B}(\mathbb{R})$) is called a *random variable* on $(A, \mathcal{A}, P)$. The *distribution or law* of ξ is the Borel probability measure $P \circ \xi$ on $\mathbb{R}$ defined by

$$P \circ \xi \; (B) = \; P\,(\xi^{-1}\,(B)), \quad B \in \mathcal{B}(\mathbb{R}). \tag{1.1.3}$$

Let Y be a real separable Banach space. For any Borel measure ν on Y we can define its *characteristic function* (or Fourier transform) $\hat{\nu}$ as follows

$$\hat{\nu}(\eta) \; = \; \int_Y e^{i<y,\eta>} \, \nu(dy), \;\; \eta \in Y', \tag{1.1.4}$$

where Y' stands for the *topological dual* of Y and the brackets $< \cdot, \cdot >$ denote the duality pairing between Y and Y'.

We mention the following result: suppose that μ and ν are two Borel probability measures on Y such that $\hat{\mu} = \hat{\nu}$ on Y; then one has $\mu = \nu$ on $\mathcal{B}(Y)$ (see for instance Vakhania et al [81, §IV.2.1]).

1.1.1 Gaussian measures on Banach spaces

Let us notice that in infinite dimensions we do not have an analogous of the Lebesgue measure. More precisely, consider any infinite dimensional Hilbert space H, then does not exist a Borel measure on H that satisfies the following two conditions: (a) it assigns finite values to bounded Borel sets and is not degenerate; (b) it is translation invariant (see chapter I of Kuo [50] for more details).

However the Gaussian measures make sense in infinite dimensional spaces.

A Gaussian measure μ on $\mathbb{R}^n$ is determined by an element $m \in \mathbb{R}^n$ and by a linear, symmetric and non negative operator Q on $\mathbb{R}^n$. μ is the unique Borel probability measure on $\mathbb{R}^n$ such that its characteristic functions $\hat{\mu}$ is given by

$$\hat{\mu}(z) \; = \; \int_{\mathbb{R}^n} e^{i<x,z>} \, \mu(dx) \; = \; e^{i<m,z>} \, e^{-\frac{1}{2}<Qz,z>}, \;\; z \in \mathbb{R}^n.$$

We say that μ has mean m and covariance operator Q. If $Q = 0$ then $\mu = \delta_m$, where δ_m denotes the Dirac measure concentrated on m. If Q is positive then μ is not degenerate and has the following density, with respect to the Lebesgue measure,

$$\frac{1}{\sqrt{(2\pi)^n \det(Q)}}\, e^{-\frac{1}{2} < Q^{-1}(x-m),(x-m) >}, \quad x \in \mathbb{R}^n. \tag{1.1.5}$$

We will denote μ by $\mathcal{N}(m, Q)$.

Let $(X, \|\cdot\|_X)$ be a real separable Banach space. A Borel probability measure ν on X is said to be a **Gaussian measure** if any $\eta \in X'$, when considered as a random variable on $(X, \mathcal{B}(X), \nu)$, is normally distributed (i.e. its distribution is a Gaussian measure on $\mathbb{R}$). A Gaussian measure ν on X is said to be *symmetric* if any $\eta \in X'$ has a Gaussian distribution with mean $0 \in \mathbb{R}$.

In Kuelbs [49] (see also Kuo [50, §3.1]) it is proved that for any non degenerate symmetric Gaussian measure ν on X there exists a Hilbert space $(H_0, \|\cdot\|_{H_0})$ continuously and densely embedded into X, i.e.

$$X' \hookrightarrow H_0' \simeq H_0 \hookrightarrow X \ \text{ continuously and densely} \tag{1.1.6}$$

(we have identified H_0 with H_0'), such that any $\eta \in X'$, considered as a random variable on $(X, \mathcal{B}(X), \nu)$, has Gaussian distribution $\mathcal{N}(0, \|\eta\|_{H_0}^2)$. Note that H_0 is unique up to isometries. The immage of H_0 in X is called the *reproducing kernel space* of ν.

Following the Gross terminology (see Gross [40], [41]) the triple (X, H_0, ν) is said to be an *abstract Wiener space.* We mention the following result.

Theorem 1.1.1 *Let $(X, \|\cdot\|_X)$ be a real separable Banach space. Then there exists a non degenerate symmetric Gaussian measure on X.*

For the proof see for instance Kuo [50, Chapter I, Theorem 4.4] or Vakhania et al [81, page 215].

Let us consider an abstract Wiener space (X, H_0, ν). Set $\nu = p_1$ and define the family of Gaussian measures $(p_t)_{t>0}$ on $\mathcal{B}(X)$,

$$p_t(B) \ = \ p_1(\frac{B}{\sqrt{t}}), \qquad B \in \mathcal{B}(X).$$

It is easy to verify that for any $t > 0$, p_t is non degenerate and each $l \in X'$ is normally distributed with mean 0 and covariance $t\|l\|_{H_0}^2$ with respect to p_t. The measure p_t is said to be the *Wiener measure* of variance parameter t (we refer to Gross [41] and Kuo [50] for a detailed exposition of the subject).

Moreover let us notice that it holds: $p_t * p_s = p_{t+s}$, $t, s \geq 0$; here $p_t * p_s$ stands for the *convolution measure* between p_t and p_s, defined as follows

$$p_t * p_s(B) \stackrel{\text{def}}{=} p_t \otimes p_s\left(\{(x,y) \in X \times X : \ x + y \in B\}\right), \ B \in \mathcal{B}(X). \tag{1.1.7}$$

For any p_t, $t > 0$, we can consider any $l \in X'$ as a random variable on $(X, \mathcal{B}(X), p_t)$. Hence we have a linear map: $R^t : X' \to L^2(X, p_t)$. If X' is equipped with the norm inherited from H_0, then R^t is an isometry (up to the constant $\sqrt{t}$). Therefore by (1.1.6) this map extends uniquely to an isometry, denoted again by R^t,

$$h \;\mapsto\; R_h^t, \quad h \in H_0, \tag{1.1.8}$$

from H_0 into $L^2(X, p_t)$. Thanks to the isometry R^t, each $h \in H_0$ is a *Gaussian random variable* on (X, p_t) (i.e. h has a Gaussian distribution with mean 0 and covariance $t\|h\|^2_{H_0}$, $t > 0$).

Notice that R_h^t is defined p_t - a.e. on X for any $h \in H_0 \backslash X'$ and depends significantly on t, since p_t and p_s are *singular* for $t \neq s$.

We define for any $x \in X$, the Gaussian measure $p_t(x, \cdot)$, $p_t(x, B) = p_t(B - x)$, $B \in \mathcal{B}(X)$. The Feldman-Hajek theorem (see Kuo [50, Theorem II.1.2]) is stated below.

Theorem 1.1.2 *For any $t > 0$, the Gaussian measures $p_t = p_t(0, \cdot)$ and $p_t(z, \cdot)$, $z \in E$ are either equivalent or singular. They are equivalent if and only if $z \in H_0$; moreover if $h \in H_0$, the Radon-Nikodym derivative of $p_t(h, \cdot)$, with respect to p_t is given by the Cameron-Martin formula:*

$$\frac{dp_t(h, \cdot)}{dp_t}(x) \;=\; \exp\Big[-\frac{1}{2t}\|h\|^2_{H_0} \;+\; \frac{1}{t}\, R_h^t(x) \Big], \quad x \in X, \;\; p_t - \text{a.e.}. \tag{1.1.9}$$

1.1.2 Gaussian measures on Hilbert spaces

Let H be a real separable Hilbert space, with scalar product $< \cdot, \cdot >$ and norm $|\cdot|$. We first review Hilbert-Schmidt and trace class operators on H that will be frequently used (for more details on this subject we refer to Ringrose [71]).

We denote by $\mathcal{L}(H)$, the Banach space of all bounded linear operators on H, endowed with the *operator norm*

$$\|T\|_{\mathcal{L}(H)} = \sup_{|u| \le 1} |Tu| \qquad T \in \mathcal{L}(H). \tag{1.1.10}$$

Then we define $\mathcal{L}_2(H) = \{ S \in \mathcal{L}(H)$ such that for an orthonormal basis (e_k) of H we have that $\sum_{k=1}^{\infty} |Se_k|^2 = c < \infty \}$.
If $S \in \mathcal{L}_2(H)$, one can verify that $\sum_{k=1}^{\infty} |Se_k|^2$ is independent of the basis (e_k). Moreover we set, for any U, $S \in \mathcal{L}_2(H)$,

$$\|S\|_2 \;=\; \Big(\sum_{k=1}^{\infty} |Se_k|^2 \Big)^{1/2}, \quad < U, S >_{\mathcal{L}_2(H)} = \sum_{k=1}^{\infty} < Ue_k, Se_k > .$$

$\mathcal{L}_2(H)$ is called the space of all *Hilbert-Schmidt operators* on H. It is a real separable Hilbert space, equipped with the scalar product $< \cdot, \cdot >_{\mathcal{L}_2(H)}$. We introduce $\mathcal{L}_1(H) = \{ T \in \mathcal{L}(H)$ such that there exist two sequences (a_k), $(b_k) \subset H$ such that $Tx = \sum_{k=1}^{\infty} a_k < x, b_k >$, $x \in H$, and $\sum_{k=1}^{\infty} |a_k|\,|b_k| < \infty \}$.

For any $T \in \mathcal{L}_1(H)$, we set

$$\|T\|_1 = \inf \Big\{ \sum_{k=1}^{\infty} |a_k|\,|b_k| : \;\; Tx = \sum_{k=1}^{\infty} a_k < x, b_k >, \, x \in H \Big\}.$$

$\mathcal{L}_1(H)$ is called the space of all *trace class operators* on H. It is a real separable Banach space, equipped with the norm $\|\cdot\|_1$. Let (e_k) be an orthonormal basis of H, for any $T \in \mathcal{L}_1(H)$ there exists finite the trace of T with respect to (e_k): Tr (T) $= \sum_{k=1}^{\infty} < Te_k, e_k >$. One shows that Tr (T) is independent of the basis (e_k).

One can prove that if $T \in \mathcal{L}(H)$, then T is of trace class if and only if Tr $(\sqrt{T^*T}) < \infty$ (T^* denotes the *adjoint* of T). Moreover $\|V\|_1 =$ Tr $(\sqrt{V^*V})$, $V \in \mathcal{L}_1(H)$.

Some useful properties of trace class and Hilbert-Schmidt operators are listed below, without proof.

It holds: $\mathcal{L}_1(H) \subset \mathcal{L}_2(H) \subset \mathcal{L}(H)$ (with continuous embeddings).
If $T \in \mathcal{L}(H)$ is symmetric and non negative one can verify that $T \in \mathcal{L}_1(H)$ if and only if Tr $(T) < \infty$, with respect to an orthonormal basis (f_k) of H. In this case we also have Tr $(T) = \|T\|_1$.

Let now $S, T \in \mathcal{L}_2(H)$. One proves that TS and $ST \in \mathcal{L}_1(H)$; moreover $\|ST\|_1 \leq \|S\|_2 \ \|T\|_2$ and Tr $(ST) =$ Tr (TS).

For any $T \in \mathcal{L}_1(H)$, $U \in \mathcal{L}_2(H)$ one has that $T^* \in \mathcal{L}_1(H)$ and $U^* \in \mathcal{L}_2(H)$, where U^* and T^* stand for the adjoints of U and T. Moreover one has:
$\|T\|_{\mathcal{L}_1(H)} = \|T^*\|_{\mathcal{L}_1(H)}$ and $\|U\|_{\mathcal{L}_2(H)} = \|U^*\|_{\mathcal{L}_2(H)}$.

Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}_i(H)$, $i = 1, 2$. Then we have that AB and $BA \in \mathcal{L}_i(H)$ and further

$$\|AB\|_{\mathcal{L}_i(H)} \leq \|A\|_{\mathcal{L}(H)}\|B\|_{\mathcal{L}_i(H)}, \quad \|BA\|_{\mathcal{L}_i(H)} \leq \|A\|_{\mathcal{L}(H)}\|B\|_{\mathcal{L}_i(H)}, \ i = 1, 2.$$

Let $Q \in \mathcal{L}_1(H)$ and $A \in \mathcal{L}(H)$, then it holds Tr $(QA) =$ Tr (AQ).

Denote by $\mathcal{F}$ the space of all finite rank operators in $\mathcal{L}(H)$. Now we can state the following useful criterion (see Lemma 14, page 1098 of Dunford and Schwartz [29]):

Lemma 1.1.3 *Let $T \in \mathcal{L}(H)$, there results:*
(i) $T \in \mathcal{L}_2(H)$ *if and only if* $\sup\{|\text{Tr}\,(NT)|, \quad N \in \mathcal{F}$ *and* $\|N\|_{\mathcal{L}_2(H)} \leq 1\}$ $= C < \infty$; *if* $T \in \mathcal{L}_2(H)$ *then* $\|T\|_2 = C$;
(ii) $T \in \mathcal{L}_1(H)$ *if and only if* $\sup\{|\text{Tr}\,(NT)|, \quad N \in \mathcal{F}$ *and* $\|N\|_{\mathcal{L}(H)} \leq 1\}$ $= c < \infty$; *if* $T \in \mathcal{L}_1(H)$ *then* $\|T\|_1 = c$.

Given a Gaussian measure μ on H, we can define its *mean* m and its *covariance operator* Q as follows:

$$\begin{aligned}
&< m, h >= \int_H < h, x > \ \mu(dx), \\
&< Qh, k >= \int_H < h, x >< k, x > \ \mu(dx) - < h, m >< k, m >, \quad h,\, k \in H.
\end{aligned} \tag{1.1.11}$$

It turns out that Q is a symmetric non negative bounded linear operator on H. Moreover one can prove that Q is a trace class operator.

By (1.1.11) it follows that the characteristic function of μ is $\hat{\mu}(z) = e^{i<m,z>}$ $e^{-\frac{1}{2}<Qz,z>}$, $z \in H$. Hence μ is uniquely determined by m and Q; we shall denote μ by $\mathcal{N}(m, Q)$.

Theorem 1.1.4 *Let Q be any symmetric non negative trace class operator on H and $m \in H$. Then there exists a unique Gaussian measure ν with mean m and covariance operator Q.*

One can compute many integrals with respect to $\mathcal{N}(m, Q)$ as for instance

$$\int_H |y|^2 \, \mathcal{N}(0, Q) dy \;=\; \mathrm{Tr}\,(Q). \tag{1.1.12}$$

Let now K be another real Hilbert space and $F : H \to K$ be defined as $F(h) = Lh + a, \quad h \in H$, where $L \in \mathcal{L}(H, K)$ and $a \in K$. Denoting by T^* the adjoint of T and using (see (1.1.3)), there results

$$F \circ \mathcal{N}(m, Q) \;=\; \mathcal{N}(Tm + a, \, TQT^*), \tag{1.1.13}$$

The proof of the next result can be found in Parthasarathy [61].

Proposition 1.1.5 *Let $(\mathcal{N}(0, Q_j))$, $j \geq 1$, and $\mathcal{N}(0, Q)$ be Gaussian measures on H. If $\lim_{j\to\infty} \|Q - Q_j\|_{\mathcal{L}_1(H)} = 0$, then $\mathcal{N}(0, Q_j)$ converges weakly to $\mathcal{N}(0, Q)$, see (1.1.1).*

Let us notice that a Gaussian measure $\mathcal{N}(m, Q)$ on H is not degenerate if and only if Q is positive (i.e. Q is non negative and one to one).

Consider a non degenerate Gaussian measure $\nu = \mathcal{N}(0, Q)$. We set $H_0 \;=\; Q^{1/2}H$ and define the following scalar product:

$$< x, y >_{H_0} \stackrel{\mathrm{def}}{=} < Q^{-1/2}x, Q^{-1/2}y >_H, \quad x, \, y \in H_0. \tag{1.1.14}$$

It is easy to see that $(H_0, < \cdot, \cdot >_{H_0}, | \cdot |_{H_0})$ is a real separable Hilbert space and $Q^{1/2} : H \;\to H_0$, turns out to be a linear and onto isometry.

Moreover H_0 is densely and continuously embedded in H, with respect to the natural embedding. Notice that it holds:

$$|Q^{1/2}h| \;\leq\; \|Q^{1/2}\|_{\mathcal{L}(H)}|h| \;=\; \|Q^{1/2}\|_{\mathcal{L}(H)}\,|Q^{1/2}h|_{H_0}, \quad h \in H_0. \tag{1.1.15}$$

It is possible to show that H_0 is the reproducing kernel space of the Gaussian measure ν. First let us remark that $H' \hookrightarrow H_0' \simeq H_0 \hookrightarrow H$ (continuously and densely).

By using (1.1.11) we obtain readily that each $h' \in H'$ is normally distributed with mean 0 and covariance $|h'|^2_{H_0}$.

Therefore (H, H_0, ν) is an abstract Wiener space.

We have $p_t = \mathcal{N}(0, tQ)$, $t > 0$. In Theorem 2.21 of Da Prato and Zabczyk [23] it is proved a special case of (1.1.9): if $x = Q^{1/2}h, \; h \in H$, the Radon-Nikodym derivative of $\mathcal{N}(Q^{1/2}h, Q)$ with respect to $\mathcal{N}(0, Q)$, is given by

$$\frac{d\mathcal{N}(Q^{1/2}h, Q)}{d\mathcal{N}(0, Q)}(y) = \exp\,[-\frac{1}{2}|h|^2 + < h, Q^{-1/2}y >], \;\; y \in H, \; \mathcal{N}(0, Q) - a.e., \tag{1.1.16}$$

where the random variable $< h, Q^{-1/2}(\cdot) >$ is defined as follows

$$< h, Q^{-1/2}y > \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{< h, e_k >< y, e_k >}{\sqrt{\lambda_k}}, \quad y \in H, \ \mathcal{N}(0,Q) - \text{ a.e. }, \tag{1.1.17}$$

where $\{e_k, \lambda_k\}$ is an eigensequence associated with Q and the series converges in $L^2(H, \mathcal{N}(0,Q))$. By (1.1.16), replacing Q with tQ, we obtain for any $h_0 \in H_0$, $t > 0$, $y \in H$ $\mathcal{N}(0,tQ)$ - a. e.,

$$\frac{d\mathcal{N}(h_0, tQ)}{d\mathcal{N}(0, tQ)}(y) = \exp\left[- \frac{1}{2t}|Q^{-1/2}h_0|^2 + \frac{1}{\sqrt{t}} < Q^{-1/2}h_0, (tQ)^{-1/2}y > \right]. \tag{1.1.18}$$

Comparing (1.1.9) and (1.1.18) we obtain that, for any $h_0 \in H_0$,

$$R^t_{h_0}(y) \ = \ \sqrt{t} \ < Q^{-1/2}h_0, (tQ)^{-1/2}y >, \quad y \in H, \ \mathcal{N}(0,tQ) - a.e.. \tag{1.1.19}$$

1.2 Differentiability in infinite dimensions

In this section $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two real Banach spaces. We denote by $\mathcal{L}(X,Y)$, the Banach space of all linear and continuous operators from X to Y endowed with the *operator norm*

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|u\| \le 1} \|Tu\|_Y \qquad T \in \mathcal{L}(X,Y). \tag{1.2.1}$$

In case when $X = Y$, we set $\mathcal{L}(X,Y) = \mathcal{L}(X)$. Moreover if $Y = \mathbb{R}$, we set $\mathcal{L}(X,\mathbb{R}) = X'$ (the *topological dual* of X).

We shall also use on $\mathcal{L}(X,Y)$ the *strong topology*, which we define by using nets. A net $(T_i : i \in I)$ in $\mathcal{L}(X,Y)$ converges to $T \in \mathcal{L}(X,Y)$, with respect to the strong topology, if for any $v \in X$, $\lim_{i\in I} T_i(v) = T(v)$ in Y. The strong topology is a locally convex topology on X (see for instance Yosida [88, §IV.7] for details). We denote by $\mathcal{L}_s(X,Y)$, the space $\mathcal{L}(X,Y)$, endowed with the strong topology. Notice that the strong topology in X' coincides with the one $\sigma(X,X')$, the weakest topology on X that makes every $l \in X'$ continuous on X.

Let Ω be an open subset of X. Next we introduce three different types of differentiability.

A map $f : \Omega \to Y$ is said to be *Gâteaux differentiable* at the point $x \in \Omega$ if there exists $Df(x) \in \mathcal{L}(X,Y)$ such that:

$$\lim_{s \to 0^+} \frac{f(x+sv) - f(x)}{s} \ = \ Df(x)(v), \quad v \in X. \tag{1.2.2}$$

Moreover if for any compact set $K \subset X$ (resp. bounded set $B \subset X$) the limit in (1.2.2) is uniform in $v \in K$ (resp. in $v \in B$), then f is said to be *Hadamard differentiable* at $x \in \Omega$ (resp. *Fréchet differentiable* at $x \in \Omega$) and $Df(x)$ is called *the Hadamard* (resp. *Fréchet*) *derivative* of f at x. Moreover we say that the map

$f : \Omega \to Y$ is Fréchet (resp. Gâteaux or Hadamard) differentiable on Ω if it is Fréchet (resp. Gâteaux or Hadamard) differentiable at any $x \in \Omega$.

Of course if X has finite dimension then Hadamard and Fréchet differentiability coincide. We remark that the Hadamard derivative is rarely considered elsewhere despite its many advantages. To this purpose we present the following two results without proof (for a detailed exposition of the Hadamard differentiability we refer to Flett [34]).

Proposition 1.2.1 *Let X and Y be two real Banach spaces and let Ω be an open subset of X. If a map $f : \Omega \to Y$ is Hadamard differentiable at $x \in \Omega$ then f is also continuous at x.*

Proposition 1.2.2 *Let X, Y and G be three real Banach spaces and let Ω_1 and Ω_2 be open subsets respectively of X and Y. Consider a map $f : \Omega_1 \to Y$ (resp. $g : \Omega_2 \to G$) that is Hadamard differentiable at $x \in \Omega_1$ (resp. at $y = f(x) \in \Omega_2$). Then the map $g \circ f$ is Hadamard differentiable at x with the Hadamard derivative $\hat{D}(g \circ f)(x) = \hat{D}g(y) \circ \hat{D}f(x)$ (here $\hat{D}g(y)$ and $\hat{D}f(x)$ stand for the Hadamard derivatives).*

Let us point out that the above two propositions do not hold when the Hadamard differentiability is replaced by the Gâteaux differentiability.

$\mathcal{L}^n(X)$, $n \geq 2$, denotes the Banach space of all n-linear and bounded functionals $A : X \times X \ldots \times X$ (n- times) $\to \mathbb{R}$, endowed with the norm

$$\|A\|_{\mathcal{L}^n} = \sup_{\|u_1\|_X = 1, \ldots \|u_n\|_X = 1} |A(u_1, \ldots, u_n)|, \quad A \in \mathcal{L}^n(X). \tag{1.2.3}$$

Of course $\mathcal{L}(X, X')$ is isometrically isomorphic to $\mathcal{L}^2(X)$.

Consider a map $f : \Omega \to \mathbb{R}$ which is Fréchet differentiable on Ω. If the Fréchet derivative $Df : \Omega \to X'$ is again Fréchet differentiable at $x \in \Omega$, we denote by $D^2 f(x)$, the Fréchet derivative of Df at x and call it the second Fréchet derivative of f at x. Clearly $D^2 f(x) \in \mathcal{L}^2(X)$. Inductively one can define the Fréchet derivatives $D^k f(x) \in \mathcal{L}^k(X)$ of any order k, $k \geq 2$.

A map $f : \Omega \to \mathbb{R}$ is said to be *Fréchet differentiable up to the order $k \geq 2$* on Ω if there exist the Fréchet derivatives $Df(x)$, ... ,$D^k f(x)$ for any $x \in \Omega$. Of course one can also consider the Gâteaux or Hadamard derivatives of order $k \geq 2$ for f.

In the sequel we will also deal with differentiability along a subspace. This concept is used by Gross, Kuo and many others in the theory of abstract Wiener spaces (see for instance Elson [32], Goodman [37], Gross [40], [41], Kuo [50], Lee [53], Piech [62], [63], [64]).

Let $(H_0, |\cdot|_{H_0})$ be a real Hilbert space such that

$$H_0 \subset X \text{ (i.e. } H_0 \text{ is continuously embedded in } X). \tag{1.2.4}$$

We identify H_0 with a subspace of X. Of course the H_0-norm is stronger than the X-norm. A mapping $f : X \to Y$ is said to be H_0-*Fréchet differentiable* at $x \in X$ if there exists $T_x \in \mathcal{L}(H_0, Y)$ such that:

$$f(x+h) - f(x) = T_x(h) + o(|h|_{H_0}), \quad h \in H_0. \tag{1.2.5}$$

T_x will be called the H_0-*Fréchet derivative* of f at $x \in X$. This is the original notion of differentiability along a subspace, introduced by Gross.

We define in addition the H_0-Gâteaux differentiability, see Priola [67]. A map $f : X \to Y$ is said to be H_0-*Gâteaux differentiable* at $x \in X$, if there exists $G_x \in \mathcal{L}(H_0, Y)$ such that :

$$\lim_{s \to 0^+} \frac{f(x+sh) - f(x)}{s} = G_x(h), \quad h \in H_0. \tag{1.2.6}$$

G_x will be called the H_0-*Gâteaux derivative* of f at $x \in X$. Clearly, using condition (1.2.4), if f is Fréchet (resp. Gâteaux) differentiable at $x \in X$, in the usual meaning, it is also H_0-Fréchet (resp. H_0-Gâteaux) differentiable at x and the respective derivatives coincide.

Consider a mapping $f : \Omega \to \mathbb{R}$ which is H_0-Fréchet differentiable on Ω. If the H_0-Fréchet derivative $D_{H_0} f : \Omega \to H_0'$ is again H_0-Fréchet differentiable at $x \in \Omega$, we denote by $D^2_{H_0} f(x)$, the H_0-Fréchet derivative of $D_{H_0} f$ at x and call it the *second* H_0-*Fréchet derivative* of f at x. Clearly $D^2_{H_0} f(x) \in \mathcal{L}^2(H_0)$. Inductively one can define the H_0-Fréchet derivatives $D^k_{H_0} f(x) \in \mathcal{L}^k(H_0)$ of any order k, $k \geq 2$.

A map $f : \Omega \to \mathbb{R}$ is said to be H_0-*Fréchet differentiable up to the order* $k \geq 2$ on Ω if there exist the H_0-Fréchet derivatives $D_{H_0} f(x), \dots, D^k_{H_0} f(x)$ for any $x \in \Omega$. Similarly we can also consider the Gâteaux or Hadamard H_0-derivatives of order $k \geq 2$ for f.

The next lemma is a generalization of a well known result that establishes a connection between Fréchet and Gâteaux differentiability.

Lemma 1.2.3 *Let X, Y be real Banach spaces and let H_0 be a real Hilbert space such that $H_0 \subset X$. Let $f : X \to Y$ be a continuous and H_0-Gâteaux differentiable mapping at each $x \in X$. Denote by $D_G f(x)$ the H_0-Gâteaux derivative of f at $x \in X$. Suppose further that $D_G f : X \to \mathcal{L}(H_0, Y)$ is continuous.*

Then f is H_0-Fréchet differentiable at each $x \in X$ and its H_0-Fréchet derivative at $x \in X$ coincides with $D_G f(x)$.

Proof Fix $\hat{x} \in X$. We introduce a map $g : H_0 \to Y$, $g(h) = f(\hat{x} + h)$, $h \in H_0$. The map g is continuous on H_0, since f is continuous and (1.2.4) holds.

Clearly the assertion is equivalent to prove that g is Fréchet differentiable, in the usual meaning, in $h = 0$ with the Fréchet derivative $Dg(0) = D_G f(\hat{x})$. Let us remark that from (1.2.6), for any $h \in H_0$, we have

$$\lim_{s \to 0^+} \frac{g(h+sk) - g(h)}{s} = D_G f(\hat{x}+h)(k), \quad k \in H_0,$$

so that g is Gâteaux differentiable at each $h \in H_0$ and has $D_G f(\hat{x}+h)$ as Gâteaux derivative at $h \in H_0$.

By our assumptions, the map $x \mapsto D_G f(x)$ is continuous from X into $\mathcal{L}(H_0, Y)$. Thus, using (1.2.4), also the map: $H_0 \to \mathcal{L}(H_0, Y)$, $h \mapsto D_G f(\hat{x}+h)$ is continuous. By a well known result, g is Fréchet differentiable at 0 and further the Frèchet derivative $Dg(0) = D_G f(\hat{x})$. The proof is complete. ■

1.3 Main functions spaces

We present here some functions spaces that will be used in the sequel. Throughout this section $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ will be two real Banach spaces.

Let $S \subset X$, $\mathcal{C}_b(S,Y)$ denotes the Banach space of all uniformly continuous and bounded maps from S into Y, endowed with the sup norm:

$$\|f\|_0 \stackrel{\text{def}}{=} \sup_{x\in S} \|f(x)\|_Y, \qquad f \in \mathcal{C}_b(S,Y).$$

In order to emphasize the space Y, sometimes we will write

$$\|f\|_{0,Y} \quad \text{instead of} \quad \|f\|_0, \quad f \in \mathcal{C}_b(S,Y). \tag{1.3.1}$$

If $Y = \mathbb{R}$, we set $\mathcal{C}_b(S,\mathbb{R}) = \mathcal{C}_b(S)$. This convention will be used for other functions spaces as well.

For any $f \in \mathcal{C}_b(S,Y)$, we denote by $\omega_f : [0,\infty) \to [0,\infty)$, the modulus of continuity of f, i.e.

$$\omega_f(r) \;=\; \sup_{x,y\in S,\, \|x-y\|_X \le r} \|f(x)-f(y)\|_Y, \quad r \ge 0.$$

Let now Ω be an open subset of X, we will consider the following subspaces of $\mathcal{C}_b(\Omega,Y)$, for $\theta \in (0,1)$,

$\mathcal{C}_b^\theta(\Omega,Y) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_b(\Omega,Y),\;$ such that $[f]_\theta \stackrel{\text{def}}{=} \sup_{x,y\in\Omega, x\neq y} \frac{\|f(x)-f(y)\|_Y}{\|x-y\|_X^\theta} < \infty\}$;

$\mathcal{C}_b^{0,1}(\Omega,Y) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_b(\Omega,Y),\;$ such that $\mathrm{Lip}(f) \stackrel{\text{def}}{=} \sup_{x,y\in\Omega, x\neq y} \frac{\|f(x)-f(y)\|_Y}{\|x-y\|_X} < \infty\}$;

$\mathcal{C}_b^1(\Omega,Y) \stackrel{\text{def}}{=} \{\, f \in \mathcal{C}_b(\Omega,Y)$, Fréchet differentiable on Ω and having the Fréchet derivative $Df \in \mathcal{C}_b(\Omega, \mathcal{L}(X,Y))\,\}$;

$\mathcal{C}_b^{1,1}(\Omega) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_b^1(\Omega)$, having the Fréchet derivative $Df \in \mathcal{C}_b^{0,1}(\Omega, X')\,\}$;

$\mathcal{C}_b^k(\Omega) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_b^1(\Omega)$, Fréchet differentiable up to the order k on Ω and having the Fréchet derivatives $D^i f \in \mathcal{C}_b(\Omega, \mathcal{L}^i(X))$, $1 \le i \le k\,\}$, $\quad k \ge 1$.

We set $\mathcal{C}_b^\infty(\Omega) = \cap_{n\ge 1} \mathcal{C}_b^n(\Omega)$. We point out that $\mathcal{C}_b^\theta(\Omega,Y)$ and $\mathcal{C}_b^{0,1}(\Omega,Y)$ are Banach spaces, respectively equipped with the norm

$$\|f\|_{0,1} = \|f\|_0 + \mathrm{Lip}\,(f), \;\; \|g\|_\theta = \|g\|_0 + [g]_\theta, \;\; f \in \mathcal{C}_b^{0,1}(\Omega,Y),\; g \in \mathcal{C}_b^\theta(\Omega,Y).$$

The space $\mathcal{C}_b^k(\Omega)$ is a Banach space, endowed with the norm

$$\|f\|_k \;=\; \|f\|_0 \;+\; \sum_{j=1}^k \|D^j f\|_0, \quad f \in \mathcal{C}_b^k(\Omega), \quad k \ge 1.$$

We use the Hadamard differentiabilty in order to define the following spaces

$\mathcal{C}^1_s(\Omega, Y) \stackrel{\text{def}}{=} \{ \ f \in \mathcal{C}_b(\Omega, Y)$, Hadamard differentiable on Ω, having the Hadamard derivative $\hat{D}f$ such that, for any $u \in X$, the map $y \mapsto \hat{D}f(y)(u)$ belongs to $\mathcal{C}_b(\Omega, Y)\}$,

$\mathcal{C}^2_s(\Omega) \stackrel{\text{def}}{=} \{f \in \mathcal{C}^{1,1}_b(\Omega)$, having the second Hadamard derivative $\hat{D}^2 f$ on Ω and such that, for any $z \in X$, the map $y \mapsto \hat{D}^2 f(y)(z)$ belongs to $\mathcal{C}_b(\Omega, X')$ $\}$.

In Chapter 2 we will prove some density theorems by using $\mathcal{C}^2_s(\Omega)$ and $\mathcal{C}^1_s(\Omega, Y)$. As concerns the differentiability along a subspace, for any $k \geq 1$, we consider the spaces

$\mathcal{C}^k_{H_0}(\Omega) = \{f \in \mathcal{C}_b(\Omega)$, that are H_0-Fréchet differentiable up to the order k and having the H_0-Fréchet derivatives $D^j_{H_0} f \in \mathcal{C}_b(\Omega, \mathcal{L}^j(H_0))$, $1 \leq j \leq k$ $\}$,

where $\mathcal{L}^1(H_0) = H_0$. Moreover we set $\mathcal{C}^\infty_{H_0}(\Omega) = \cap_{k\geq 1}\mathcal{C}^k_{H_0}(\Omega)$. The space $\mathcal{C}^k_{H_0}(\Omega)$ are associated with regularity properties of the heat semigroup in $\mathcal{C}_b(X)$, when (X, H_0, p_1) is an abstract Wiener space (see for instance Proposition 9 in Gross [41], §II.6 in Kuo [50] and Piech [64]). It is straightforward to verify that each $\mathcal{C}^k_{H_0}(\Omega)$ is a Banach space endowed with the norm:

$$\|f\|_{k,H_0} = \|f\|_0 + \sum_{j=1}^{k} \|D^j_{H_0} f\|_0, \quad f \in \mathcal{C}^k_{H_0}(\Omega).$$

1.3.1 The spaces $\mathcal{C}^r_Q(\Omega)$

Let H be a real separable Hilbert space, with scalar product $< \cdot, \cdot >$ and norm $|\cdot|$. Let Q be a positive symmetric trace class operator on H and Ω be an open set of H. We fix an orthonormal basis (e_k) of H that diagonalizes Q: $Qx = \sum_{k=1}^\infty \lambda_k < e_k, x >$, $x \in H$. We introduce the following spaces related to Q.

$\mathcal{C}^1_Q(\Omega)$ is the set of all $f \in \mathcal{C}_b(\Omega)$ such that:

(i) for any $v \in H$, $x \in \Omega$, there exists the derivative of f at x, in the direction $Q^{1/2}v$ that we denote by $D_{Q^{1/2}v} f(x)$;
(ii) for any $x \in \Omega$, there exists $D_Q f(x) \in H$ such that:

$$D_{Q^{1/2}v} f(x) = < D_Q f(x), v >, \quad \forall v \in H;$$

(iii) the mapping $\Omega \to H$, $x \mapsto D_Q f(x)$ belongs to $\mathcal{C}_b(\Omega, H)$.

It is easy to prove that if $f \in \mathcal{C}^1_Q(\Omega)$, defining the *partial derivatives* $D_k f = D_{e_k} f$, $k \geq 1$, we have $D_Q f(x) = \sum_{k=1}^\infty \sqrt{\lambda_k} D_k f(x) e_k$, $x \in \Omega$.

$\mathcal{C}^2_Q(\Omega)$ is the set of all functions in $\mathcal{C}^1_Q(\Omega)$ such that:

(i) for any $v \in H$, $x \in \Omega$, there exists the directional derivative

$$D_{Q^{1/2}v}\,[D_Q f](x) \;=\; \lim_{s\to 0^+} \frac{D_Q f(x+sQ^{1/2}v)-D_Q f(x)}{s} \quad \text{in} \;\; H;$$

(ii) for any $x\in\Omega$, there exists $D_Q^2 f(x)\in\mathcal{L}(H)$, such that

$$D_{Q^{1/2}v}\,[D_Q f](x) \;=\; D_Q^2 f(x)(v), \qquad v\in H;$$

(iii) the map $\Omega\to\mathcal{L}(H)$, $x\mapsto D_Q^2 f(x)$ belongs to $\mathcal{C}_b(\Omega,\mathcal{L}(H))$.

Setting $D_{e_h}(D_k f) \;=\; D_{hk}f$, $h,\,k\geq 1$, we can easily see that
$< D_Q^2 f(x)u, v >= \sum_{h,k=1}^{\infty}\sqrt{\lambda_h\lambda_k}\; D_{hk}f(x)u_k v_h$, $x\in\Omega$, $u,v\in H$, $f\in\mathcal{C}_Q^2(\Omega)$.

In a similar way it is possible to define the spaces $\mathcal{C}_Q^n(\Omega)$ with the differential operators D_Q^n, $n\geq 1$, and also $\mathcal{C}_Q^\infty(\Omega) \;=\; \cap_{n\geq 1}\,\mathcal{C}_Q^n(\Omega)$.
Every $\mathcal{C}_Q^n(\Omega)$, $n\geq 1$, turns out to be a Banach space with respect to the norm

$$\|f\|_{n,Q} \;=\; \|f\|_0 \;+\; \sum_{j=1}^{n}\; \|D_Q^j f\|_0, \qquad f\in\mathcal{C}_Q^n(\Omega).$$

$\mathcal{C}_Q^\theta(\Omega,X)$, $\theta\in(0,1)$, is the set of all functions $f\in\mathcal{C}_b(\Omega,X)$ such that there exists $M=M(\theta,Q,f)>0$ and

for any $z,w\in\Omega$ with $z-w\in Q^{1/2}H$: $\|f(z)-f(w)\|_X \;\leq M\,\|Q^{-1/2}(z-w)\|_H^\theta$.

$\mathcal{C}_Q^\theta(\Omega,X)$ is a Banach space endowed with the norm

$$\|f\|_{\theta,Q} \;=\; \|f\|_0 \;+\; [f]_{\theta,Q}, \qquad [f]_{\theta,Q} \;=\; \sup_{z,w\in\Omega\;/z-w\in Q^{1/2}H} \frac{\|f(z)-f(w)\|_X}{\|Q^{-1/2}(z-w)\|_H^{\,\theta}},$$

where $f\in\mathcal{C}_Q^\theta(\Omega,X)$. When $X=\mathbb{R}$, we set $\mathcal{C}_Q^\theta(\Omega) \;=\; \mathcal{C}_Q^\theta(\Omega,\mathbb{R})$, $\theta\in(0,1)$.
Moreover we define:

$$\mathcal{C}_Q^{1+\theta}(\Omega) \overset{\text{def}}{=} \{h\in\mathcal{C}_Q^1(\Omega) \;\;/\;\; D_Q f\in\mathcal{C}_Q^\theta(\Omega,H)\;\}$$

that is a Banach space endowed with the norm $\|h\|_{1+\theta,Q} \;=\; \|h\|_{1,Q} \;+\; \|D_Q h\|_{\theta,Q}$, $h\in\mathcal{C}_Q^{1+\theta}(\Omega)$.

Some comments on the previous spaces are in order. First let us remark that $\mathcal{C}_b^1(\Omega)\subset\mathcal{C}_Q^1(\Omega)$ and further $D_Q f \;=\; Q^{1/2}Df$ for any $f\in\mathcal{C}_b^1(\Omega)$. Clearly $\mathcal{C}_b^\theta(\Omega)\subset\mathcal{C}_Q^\theta(\Omega)$ for any $\theta\in(0,1)$.

Now we briefly discuss the spaces $\mathcal{C}_Q^\theta(\Omega)$. Our definition of $Q-$Hölder continuous functions follows Priola [66]. It concides with that previously introduced in Cannarsa and Da Prato [12], [13] in case of $\Omega=H$. This fact is shown in the next result.

Proposition 1.3.1 *We have, for any $\theta\in(0,1)$, $f\in\mathcal{C}_b(H)$,*

$$f\in\mathcal{C}_Q^\theta(H) \iff f(Q^{1/2}\,\cdot)\in\mathcal{C}_b^\theta(H),$$

further the Hölder constant $[f(Q^{1/2}\cdot)\,]_\theta$ concides with $[f]_{\theta,Q}$.

Proof Of course if $g \in \mathcal{C}^\theta_Q(H)$ then $g(Q^{1/2}\cdot) \in \mathcal{C}^\theta_b(H)$ and $[g(Q^{1/2}\cdot)]_\theta \le [g]_{\theta,Q}$. Let us prove the converse implication.

Let $f \in \mathcal{C}_b(H)$ such that $f(Q^{1/2}\cdot) \in \mathcal{C}^\theta_b(H)$. Fix $x,\, y \in H$ such that $x-y \in Q^{1/2}H$.

We set $h = Q^{-1/2}(x-y)$ so that $x = Q^{1/2}h + y$. Since $Q^{1/2}H$ is dense in H, we can choose a sequence $(Q^{1/2}y_n) \subset Q^{1/2}H$ which converges to y. Now by hypothesis one has

$$\begin{aligned}|f(Q^{1/2}y_n + Q^{1/2}h) - f(Q^{1/2}y_n)| &= |f(Q^{1/2}[y_n - h]) - f(Q^{1/2}y_n)|\\ &\le [f(Q^{1/2}\cdot)]_\theta\, |h|^\theta, \quad n \ge 1.\end{aligned}$$

Letting $n \to \infty$ we obtain

$$|f(y + Q^{1/2}h) - f(y)| \le [f(Q^{1/2}\cdot)]_\theta\, |h|^\theta.$$

Thanks to the arbitrariness of x and y we conclude that $f \in \mathcal{C}^\theta_Q(H)$ and further that $[f]_{\theta,Q} \le [f(Q^{1/2}\cdot)]_\theta$. The proof is complete. ■

The space $\mathcal{C}^1_Q(\Omega)$ are introduced in [12] in case when $\Omega = H$. The spaces $\mathcal{C}^n_Q(\Omega)$, $n \ge 2$, are introduced in Priola [66]; they are a slight modification of those considered by Cannarsa and Da Prato [12]. Another equivalent definition for the spaces $\mathcal{C}^n_Q(H)$ is given in Zambotti [86].

As in § 1.2 we consider $H_0 = Q^{1/2}H$, the reproducing kernel space of the Gaussian measure $\mathcal{N}(0,Q)$. We are going to prove the following result, see Priola [67].

Proposition 1.3.2 *Let $(H, <\cdot,\cdot>, |\cdot|)$ be a real separable Hilbert space and let Q be a positive symmetric trace class operator on H. Consider the Hilbert space $H_0 = Q^{1/2}H$ with respect to the inner product $<x,y>_{H_0} = <Q^{-1/2}x, Q^{-1/2}y>$, $x,y \in H_0$. Then for any open set $\Omega \subset H$ we have $\mathcal{C}^n_Q(\Omega) = \mathcal{C}^n_{H_0}(\Omega)$, $n \ge 1$.*

First it is clear that $Q^{1/2} : H \to H_0$ turns out to be a linear and onto isometry. It follows that, for any $n \ge 1$, $\mathcal{L}^n(H)$ is isometrically isomorphic to $\mathcal{L}^n(H_0)$ by means of the linear isometry $T_n : \mathcal{L}^n(H) \to \mathcal{L}^n(H_0)$,

$$T_n(A)(v_1,\dots,v_n) = A(Q^{-1/2}v_1,\dots,Q^{-1/2}v_n),\ A \in \mathcal{L}^n(H),\ v_1,\dots,v_n \in H_0. \quad (1.3.2)$$

The proof of this fact is simple, we only consider $n = 2$. We have $T_2(A) = Q^{1/2}AQ^{-1/2}$, $A \in \mathcal{L}(H)$ and consequently

$$<T_2(A)x, y>_{H_0} = <AQ^{-1/2}x, Q^{-1/2}y>, \quad x,\, y \in H_0, \quad (1.3.3)$$

so that $|<T_2(A)x,y>_{H_0}| \le \|A\|_{\mathcal{L}} |Q^{-1/2}x|\,|Q^{-1/2}y| = \|A\|_{\mathcal{L}} |x|_{H_0} |y|_{H_0}$, $x,y \in H_0$ and so the continuity of $T_2(A)$ is clear. Moreover

$$<T_2^{-1}(B)u, v> = <BQ^{1/2}u, Q^{1/2}v>_{H_0}, \quad u,\, v \in H.$$

Proof of Proposition 1.3.2 Let us consider $n = 1$. First we prove that $\mathcal{C}^1_Q(\Omega) \subset \mathcal{C}^1_{H_0}(\Omega)$.

Fix $f \in \mathcal{C}^1_Q(\Omega)$, by definition we have for any $x \in \Omega,\ v \in H$,

$$\lim_{s\to 0^+} \frac{f(x+sQ^{1/2}v)-f(x)}{s} =< D_Q f(x), v >=< Q^{1/2}\, D_Q f(x), Q^{1/2} v >_{H_0} .$$

Thus we get that f is H_0-Gâteaux differentiable at each point $x \in \Omega$ and moreover its H_0-Gâteaux derivative is $Q^{1/2} D_Q f(x)$.

By hypothesis, the map $x \mapsto D_Q f(x)$ belongs to $\mathcal{C}_b(\Omega, H)$ and so we deduce that the map: $\Omega \to H_0,\ x \mapsto Q^{1/2} D_Q f(x)$ belongs to $\mathcal{C}_b(\Omega, H_0)$.

Now we apply Lemma 1.2.3 in order to conclude that f is also H_0-Fréchet differentiable at each $x \in \Omega$ and further its H_0-Fréchet derivative is $Q^{1/2} D_Q f(x)$. We have proved that $f \in \mathcal{C}^1_{H_0}(\Omega)$.
To verify that $\mathcal{C}^1_{H_0}(\Omega) \subset \mathcal{C}^1_Q(\Omega)$, we can proceed as in the previous part. We only remark that if $f \in \mathcal{C}^1_{H_0}(\Omega)$ we have that $D_Q f(x) = Q^{-1/2} D_{H_0} f(x),\ x \in \Omega$.

We consider $n = 2$. We use the operator $T_2 : \mathcal{L}(H) \to \mathcal{L}(H_0)$, defined in (1.3.3).

First we prove that $\mathcal{C}^2_Q(\Omega) \subset \mathcal{C}^2_{H_0}(\Omega)$.
Fix $f \in \mathcal{C}^2_Q(\Omega)$, by the above part we already know that $f \in \mathcal{C}^1_{H_0}(\Omega)$ and that its H_0-Fréchet derivative is $D_{H_0} f(x) = Q^{1/2} D_Q f(x),\ x \in \Omega$.

We claim that $D_{H_0} f : \Omega \to H_0$ admits H_0-Gâteaux derivative: $D^2_{H_0} f(x) = T_2\left(D^2_Q f(x)\right),\ x \in \Omega$. Indeed, by the assumptions on f, for any $x \in \Omega$, we have

$$\begin{aligned}
0 &= \lim_{s\to 0^+} \sup_{|u|=1} \Big| < \frac{D_Q f(x+sQ^{1/2}v) - D_Q f(x)}{s}, u > - < D^2_Q f(x)(v), u > \Big| \\
&= \lim_{s\to 0^+} \sup_{|u|=1} \Big| < \frac{D_{H_0} f(x+sQ^{1/2}v) - D_{H_0} f(x)}{s} - T_2(D^2_Q f(x))(Q^{1/2}v), Q^{1/2}u >_{H_0} \Big| \\
&= \lim_{s\to 0^+} \sup_{w\in H_0,\, |w|_{H_0}=1} \Big| < \frac{D_{H_0} f(x+sQ^{1/2}v) - D_{H_0} f(x)}{s} - T_2(D^2_Q f(x))(Q^{1/2}v), w >_{H_0} \Big|.
\end{aligned} \tag{1.3.4}$$

Since $D^2_Q f$ belongs to $\mathcal{C}_b(\Omega, \mathcal{L}(H))$, it follows that $T_2(D^2_Q f(\cdot))$ belongs to $\mathcal{C}_b(\Omega, \mathcal{L}(H_0))$. Applying again Lemma 1.2.3 we deduce that $T_2(D^2_Q f(x))$ is the second H_0-Fréchet derivative of f at $x \in \Omega$. Thus we have obtained that $f \in \mathcal{C}^2_{H_0}(\Omega)$.

The proof that $\mathcal{C}^2_{H_0}(\Omega) \subset \mathcal{C}^2_Q(\Omega)$ is similar. We only point out that if $f \in \mathcal{C}^2_{H_0}(\Omega)$, we have $D^2_Q f(x) = T_2^{-1}(D^2_{H_0} f(x)),\ x \in \Omega$.

As concerns the spaces $\mathcal{C}^n_Q(\Omega)$ and $\mathcal{C}^n_{H_0}(\Omega)$ with $n > 2$, we can proceed using the same technique, except that the computation is more notationally involved. ■

1.4 Basic concepts from Interpolation Theory

Here we briefly review some real interpolation spaces and basic results from Interpolation Theory (we refer to Triebel [78] and Lunardi [55] for more details).

Throughout this section $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ stand for two real Banach spaces such that Y is *continuously embedded* in X (we briefly write $Y \subset X$).

For any $\theta \in (0,1)$, we define the real interpolation spaces

$$(X,Y)_{\theta,\infty} \stackrel{\text{def}}{=} \{x \in X \text{ such that } [x]_{\theta,\infty} = \sup_{t>0} t^{-\theta} K(t,x) < \infty\}, \tag{1.4.1}$$

where $K(t,x) = \inf\{\|a\|_X + t\|b\|_Y : x = a+b,\ a \in X, b \in Y\}$. $(X,Y)_{\theta,\infty}$ is a Banach space endowed with the norm $\|x\|_{\theta,\infty} = \|x\|_X + [x]_{\theta,\infty}$, $x \in (X,Y)_{\theta,\infty}$. We will often use the next result.

Theorem 1.4.1 *Let X, X_1, Y, Y_1 be real Banach spaces such that $Y \subset X$ and $Y_1 \subset X_1$. Let moreover $T \in \mathcal{L}(X,X_1) \cap \mathcal{L}(Y,Y_1)$. Then one has:*
$T \in \mathcal{L}((X,Y)_{\theta,\infty}, (X_1,Y_1)_{\theta,\infty})$ and further

$$\|T\|_{\mathcal{L}((X,Y)_{\theta,\infty},(X_1,Y_1)_{\theta,\infty})} \le (\|T\|_{\mathcal{L}(X,X_1)})^{1-\theta} (\|T\|_{\mathcal{L}(Y,Y_1)})^{\theta}.$$

We also need the following *Reiteration Theorem.*

Theorem 1.4.2 *Let X, Y, E, F be real Banach spaces such that*

$$Y \subset F \subset E \subset X.$$

Assume that there exist $\alpha, \beta \in [0,1]$, $\alpha < \beta$, and two positive constants C_α and C_β such that

$$\|e\|_E \le C_\alpha \|e\|_X^{1-\alpha} \|e\|_Y^{\alpha},\ \ e \in E;\ \ \|f\|_F \le C_\beta \|f\|_X^{1-\beta} \|f\|_Y^{\beta},\ f \in F.$$

Then for any $\theta \in (0,1)$, we have, setting $\eta = (1-\theta)\alpha + \theta\beta$,

$$(X,Y)_{\eta,\infty} \subset (E,F)_{\theta,\infty}.$$

Now we consider interpolation spaces related to strongly continuous semigroups of bounded linear operators.

Let P_t be a strongly continuous semigroup on X. Let $\mathcal{A}$ be the (infinitesimal) generator of P_t. Since $\mathcal{A}$ is a closed operator we can consider $D(\mathcal{A})$ as a Banach space, endowed with the *graph norm*:

$$\|y\|_{D(\mathcal{A})} \stackrel{\text{def}}{=} \|\mathcal{A}y\|_X + \|y\|_X, \quad y \in D(\mathcal{A}).$$

We set

$$\mathcal{D}_{\mathcal{A}}(\theta,\infty) \stackrel{\text{def}}{=} (\mathcal{C}_b(H),\ D(\mathcal{A}))_{\theta,\infty}, \quad \theta \in (0,1),$$

It is noteworthy that these interpolation spaces can be completely characterized in terms of P_t:

$$f \in \mathcal{D}_{\mathcal{A}}(\theta,\infty) \iff [f]_{\theta,\mathcal{A}} \stackrel{\text{def}}{=} \sup_{t\in(0,1]} \|P_t f - f\|_0\, t^{-\theta} < \infty. \tag{1.4.2}$$

Moreover it can be shown that the norm

$$\|\cdot\|_{\theta,\mathcal{A}} = \|\cdot\|_0 + [\cdot]_{\theta,\mathcal{A}}$$

is equivalent to $\|\cdot\|_{\theta,\infty}$ in $\mathcal{D}_{\mathcal{A}}(\theta,\infty)$.

Chapter 2

Uniform approximations of uniformly continuous and bounded functions on Banach spaces

2.1 Introduction and setting of the problem

In this chapter we present some our results, see Priola [65], concerning uniform approximation of uniformly continuous and bounded functions defined on infinite dimensional spaces, by means of smoother functions. The subject has an interest in view of the treatment of PDE's with infinitely many variables (see Chapter 4 and 5).

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be real Banach spaces. We consider $\mathcal{C}_b(X,Y)$, the Banach space of all uniformly continuous and bounded maps between X and Y, endowed with the supremum norm, and other functions spaces introduced in Chapter 1. This chapter develops into three parts.

In § 2.2 we provide density theorems for some subspaces of $\mathcal{C}_b(X,Y)$, under suitable assumptions on X and Y. We briefly review some known results about uniform approximation in Banach spaces (see also the introduction of Bogachev [8]). First of all if $\dim X < \infty$ then, by using mollifiers and convolution with respect to the Lebesgue measure, it is easy to prove that $\mathcal{C}_b^\infty(X,Y)$ (the space of all functions bounded together with all their derivatives of any order) is dense in $\mathcal{C}_b(X,Y)$.

When E is infinite dimensional the situation is different. Even if $Y = \mathbb{R}$, there exist many separable Banach spaces X, for which there is a function $f_0 \in \mathcal{C}_b(X)$ that is not uniformly approximable by Fréchet differentiable functions (for instance, take $X = \mathcal{C}([0,1])$ endowed with the sup norm and $f_0(x) = \min(1, \|x\|_0)$, $x \in X$), for details see Bogachev and Shkarin [7] and Bonic and Frampton [9].

However Goodman was able to prove (see [37]) that, given a real separable Banach space X, each function $f \in \mathcal{C}_b(X)$ can be approximated by bounded Lipschitz continuous functions which are differentiable in the Hadamard sense. We shall improve this result, showing that the approximating functions, used in Goodman [37], have uniformly continuous Hadamard derivatives, in a weak sense (see Theorem 2.2.2).

When X is a Hilbert space, possibly not separable, then the situation is better. Lasry and Lions have proved (see [52]) that $\mathcal{C}_b^{1,1}(X)$ (the space of all bounded Fréchet differentiable functions, having a Lipschitz continuous and bounded Fréchet derivative) is dense in $\mathcal{C}_b(X)$. However a result of Nemirowskii and Semenov [59] implies that $\mathcal{C}_b^2(X)$ (the space of all functions in $\mathcal{C}_b^{1,1}(X)$ having a bounded, uniformly continuous second Fréchet derivative) is not dense even if X is separable. We shall improve the Lasry-Lions theorem, showing that $\mathcal{C}_s^2(X)$ (the space of all functions in $\mathcal{C}_b^{1,1}(X)$ having a weakly uniformly continuous second Hadamard derivative) is dense in $\mathcal{C}_b(X)$ (see Theorem 2.2.7).

Moreover we prove that our density results also hold in $\mathcal{C}_b(S)$, where S is a subset of X (see Theorem 2.2.19).

Approximation results for maps in $\mathcal{C}_b(X, Y)$, where Y is an infinite dimensional space, are available in the literature. For any pair of Hilbert spaces H, K, a result of Valentine (see [80] and also Tsar'kov [79]) implies that any function f in $\mathcal{C}_b(H, K)$ can be approximated by a sequence (f_n) of Lipschitz continuous and bounded functions. When H is separable, a theorem of Bogachev (see [8, §2]) implies that it is possible to choose each function f_n having a bounded Hadamard derivative in H. We shall show that each f_n can be choosen having also a weakly uniformly continuous Hadamard derivative (see Theorem 2.2.6).

In general, uniformly continuous functions from a separable Banach space E to a Hilbert space Y cannot be approximated by Lipschitz continuous functions (according to Remark 1 of Bogachev [8]). However Hölder approximations of order $1/2$ are always possible (see Minty [58]). Other technical results in specific cases (concerning for instance maps between L^p spaces, see Tsar'kov [79]) are available.

In the second part (see § 2.3), we establish a strict link between uniform approximation in $\mathcal{C}_b(X)$ by smooth functions and existence of smooth Urysohn functions on X (see Theorem 2.3.3). This problem has interest for some applications to stochastic differential equations (see § 3 of Tessitore and Zabczyk [76]).

We conclude the paper (see § 2.4), by proving an approximation result (see Theorem 2.4.2) concerning real, bounded mappings on a separable Banach space X, which are uniformly continuous with respect to a locally convex topology weaker than the norm topology. This result implies that if X is reflexive, then every real, bounded and $\sigma(X, X')$-uniformly continuous function can be approximated uniformly by functions which belong to $\mathcal{C}_b^1(X)$ (the space of all functions in $\mathcal{C}_b(X)$ having a bounded, uniformly continuous Fréchet derivative).

We recall that $\mathcal{L}(X, Y)$ stands for the Banach space of all linear, continuous operators from X to Y. It is a endowed with the usual norm $\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|u\|\le 1} \|Tu\|_Y \quad T \in \mathcal{L}(X, Y)$. We shall also consider on $\mathcal{L}(X, Y)$ the strong topology and will denote by $\mathcal{L}_s(X, Y)$, the space $\mathcal{L}(X, Y)$, endowed with the strong topology.
Let G be a Banach space, it easy to verify that a map

$$T \; : G \to \quad \mathcal{L}_s(X, Y)$$

is uniformly continuous if and only if $T(\cdot)(u) : G \to Y$ is uniformly continuous, for any $u \in X$.
Now we introduce the set $\mathcal{C}_s(G, \mathcal{L}_s(X, Y))$ of all uniformly continuous functions T from G into $\mathcal{L}_s(X, Y)$ such that

$$\|T\|_0 \ = \ \sup_{u \in G} \ \|Tu\|_{\mathcal{L}(X,Y)} \ < \infty. \tag{2.1.1}$$

In view of the Uniform Boundedness Principle, $T \in \mathcal{C}_s(G, \mathcal{L}_s(X,Y))$ if and only if:

$$T(\cdot)(u) \ \in \mathcal{C}_b(G,Y), \qquad u \in X.$$

We will need the following lemma.

Lemma 2.1.1 *Let X, G, Y be Banach spaces, then for a map $T \ : G \ \to \mathcal{L}_s(X,Y)$ the following statements are equivalent:*
(i) T belongs to $\mathcal{C}_s(G, \mathcal{L}_s(X,Y))$;
(ii) for any compact set K in X, the map $T(\cdot)(\cdot)$ belongs to $\mathcal{C}_b(G \times K, Y)$.
(iii) for any compact set K in X, the map $\sup_{u \in K} T(\cdot)(u)$ belongs to $\mathcal{C}_b(G,Y)$;
Moreover each above condition implies that the map:

$$T(\cdot)(\cdot) \ \ : \ G \times X \ \to \ Y \ \ \text{ is continuous.} \tag{2.1.2}$$

Proof We prove that (i) $\Rightarrow$ (ii).
Boundedness of $T(\cdot)(\cdot)$ is clear, so we verify uniform continuity. Fix a compact set K in X, then for any $\epsilon > 0$, there exists a finite set $L = \{v_1, \dots v_n\}$ in K such that for $v \in K$ we can find $v_k \in L$ with $\|v - v_k\|_X \le \epsilon$. Take $\delta > 0$ such that $\omega_{T(\cdot)(v_i)}(s) \ \le \epsilon, \ \ 0 \le s \le \delta, \ \ i = 1 \dots n$.
Thus for any $x, y \in G$, with $\|x - y\|_G \le \delta, \quad u, v \in K$ with $\|u - v\|_X \le \epsilon$, we can choose $v_k \in L$ such that $\|u - v_k\|_X \le \epsilon$ and we get

$$\|T(x)(u) - T(y)(v)\|_Y \ \le \ \|T(x)[u - v_k]\|_Y \ + \ \|[T(x) - T(y)](v_k)\|_Y$$

$$+ \ \|T(y)[v_k - v]\|_Y \ \le \ 2\epsilon \|T\|_0 \ + \ \epsilon.$$

Finally condition (2.1.2) follows by
$\|T(x)(u) - T(z)(v)\|_Y \le \| \ [T(x) - T(z)](u)\|_Y \ + \|T(z)[u - v]\|_Y, \ \ u, v \in X, \ x, z \in G.$
We show that (ii) $\Rightarrow$ (iii).
Fix $x, z \in G$. For any $\hat{u} \in K$, one has $-\sup_{u \in K} T(z)(u) \le -T(z)(\hat{u})$. It follows that

$$\sup_{u \in K} T(x)(u) - \sup_{u \in K} T(z)(u) = \sup_{u \in K} [T(x)(u) - \sup_{u \in K} T(z)(u)].$$

$$\le \sup_{u \in K} |T(x)(u) - T(z)(u)|.$$

Changing x with z in the last formula, we obtain

$$|\sup_{u \in K} T(x)(u) - \sup_{u \in K} T(z)(u)| \ \le \ \sup_{u \in K} |T(x)(u) - T(z)(u)|$$

and assertion (iii) follows readily. The remainder implication is obvious. ∎

We recall the following functions spaces, introduced in Chapter 1,

$\mathcal{C}_s^1(X,Y) \stackrel{\text{def}}{=} \{\ f \in \mathcal{C}_b^{0,1}(X,Y)$, Hadamard differentiable on X, having the Hadamard derivative $Df \in \mathcal{C}_s(X, \mathcal{L}_s(X,Y))\ \}$,

$\mathcal{C}_s^2(X) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_b^{1,1}(X)$, having the second Hadamard derivative $D^2f(x)$, at any $x \in X$, and such that $D^2 f \in \mathcal{C}_s(X, \mathcal{L}_s(X,X')\ \}$.

To prove density theorems we need to introduce the heat semigroup on abstract Wiener spaces. Let (X, H_0, p_1) be an abstract Wiener space (see Chapter 1). Gross has proved (see [41]) that if we set

$$O_t f(x) \ = \ \int_X f(x+y)\, p_t(dy), \qquad f \in \mathcal{C}_b(X),\ \ x \in X,\ t > 0, \tag{2.1.3}$$

$O_0 = I$, then O_t is a strongly continuous semigroup of bouunded linear operators on $\mathcal{C}_b(X)$. We call it the **heat semigroup** in $\mathcal{C}_b(X)$ (associated with p_1).

2.2 Density theorems

We start this section with a general lemma. The statement (a) is essentially known (see Bogachev [8, Lemma 1] and also Flett [34, §4.2]). We provide here a self-contained proof.

Lemma 2.2.1 *Let $(X,\ \|\cdot\|_X)$, $(Y,\ \|\cdot\|_Y)$ be two real Banach spaces, $D \subset X$ a dense linear subspace and let $f \in \mathcal{C}_b^{0,1}(X,Y)$. Suppose that:*
(i) for any $x \in X$, $v \in D$, there exists:

$$\lim_{s \to 0^+} \frac{f(x+sv) - f(x)}{s} \ = \ A(x,v) \in Y;$$

(ii) for any fixed $x \in X$, $A(x,\ \cdot)$ is linear from D in Y.
Then it holds:
(a) f is Hadamard differentiable on X and $\|Df(x)\|_{\mathcal{L}(X,Y)} \le \mathrm{Lip}(f)$, $x \in X$.
If moreover
(iii) the limit in (i) is uniform in $x \in X$,
then we have
(b) $f \in \mathcal{C}_s^1(X,Y)$.

Proof (a) Assume that (i) and (ii) hold and fix $x \in X$. There results

$$\Big\| \frac{f(x+sv) - \ f(x)}{s} \Big\|_Y \le \mathrm{Lip}\ (f)\|v\|_Y,\ \ v \in D,\ s \in]0,1].$$

By this estimate it follows that $A(x,\cdot) : D \to Y$ is continuous and moreover $\|A(x,\cdot)\|_{\mathcal{L}(D,Y)} \le \mathrm{Lip}(f)$, $x \in D$.

We denote by $B(x,\cdot)$, the unique linear and continuous extension of $A(x,\cdot)$ to the all of X. Now consider $v \in X$ and take $(v_n)_{n\ge1} \subset D$ such that $v_n \to v$ as $n \to \infty$. Let us define the mappings:

$\psi_n, \ \psi \ : \ (0,1] \ \to \ Y, \qquad n \geq 1,$

$$\psi_n(s) \stackrel{\text{def}}{=} \frac{f(x+sv_n)- \ f(x)}{s}, \ \ \psi(s) \stackrel{\text{def}}{=} \frac{f(x+sv)- \ f(x)}{s}, \ \ s \in (0,1].$$

It turns out that $\psi_n \to \psi$ uniformly in $s \in (0,1]$. Indeed

$$\sup_{s\in(0,1]} \|\psi_n(s) - \psi(s)\|_Y \ \leq \ \text{Lip}(f) \ \|v_n - v\|_X \quad \to \ 0,$$

as $n \to \infty$. By hypothesis (i), there exists $\lim_{s\to 0^+} \psi_n(s) \ = A(x, v_n)$ in Y. Thus we can deduce that there exists

$$\lim_{s\to 0^+} \psi(s) \ = \lim_{n\to\infty} \ A(x, v_n) \stackrel{\text{def}}{=} B(x, v).$$

For the arbitrariness of x and v, we get the Gâteaux differentiability of f on X.

Now we denote by Df the Gâteaux derivative of f and check that f is also Hadamard differentiable on X. To this purpose we fix $x \in X$, a compact set $K \subset X$ and consider the mappings

$\eta_s : \ K \ \to Y, \quad s \in (0,1],$

$$\eta_s(v) \stackrel{\text{def}}{=} \frac{f(x+sv)- \ f(x)}{s}, \qquad v \in K.$$

We show that for any sequence $(s_n) \subset (0,1]$ such that $s_n \to 0$, there exists

$$\lim_{n\to\infty} \sup_{v\in K} \|\eta_{s_n}(v) - Df(x)(v)\|_Y \ = \ 0 \quad \text{uniformly in } v \in K. \tag{2.2.1}$$

Take any subsequence (s_n^1) of (s_n). Since f is Lipschitz continuous, $(\eta_{s_n^1})$ is an equicontinuous sequence of mappings in $\mathcal{C}_b(K, Y)$. Moreover for any $v \in K$, the sequence $\{\eta_{s_n^1}(v)\}$ is relatively compact in Y, since there exists

$$\lim_{s\to 0^+} [\, \eta_s(v) - \ Df(x)(v)\,] = 0 \ \text{ in } \ Y.$$

Applying the Arzela-Ascoli Theorem (see for instance Ash [4, § A8.5]) we can deduce that there exists a subsequence (s_n^2) of s_n^1 such that

$$\lim_{n\to\infty} \sup_{v\in K} \|\eta_{s_n^2}(v) \ - \ Df(x)(v)\|_Y \ = \ 0.$$

This way we have proved formula (2.2.1). The Hadamard differentiability at $x \in X$ is proved.

(b) Assume now that also (iii) holds. Fix $v \in X$ and take $(v_n) \subset D$, such that $v_n \to v$ as $n \to \infty$ in X. Define the maps:

$\phi_n, \ \phi : \ (0,1] \ \to \ \mathcal{C}_b(X, Y), \quad n \geq 1 \quad \text{such that}$

$$\phi_n(s) \stackrel{\text{def}}{=} \frac{f(\cdot+sv_n)- \ f(\cdot)}{s}, \ \ \phi(s) \stackrel{\text{def}}{=} \frac{f(\cdot+sv)- \ f(\cdot)}{s}, \ \ s \in (0,1]. \tag{2.2.2}$$

Arguing as for (ψ_n), we get that $\phi_n \to \phi$ uniformly in $s \in (0,1]$. Indeed

$$\lim_{n\to\infty} \sup_{s\in(0,1]} \|\phi_n(s)-\phi(s)\|_{\mathcal{C}_b(X,Y)} = \lim_{n\to\infty} \sup_{s\in(0,1]} \sup_{x\in X} \Big\|\frac{f(x+\ sv_n)-f(x+\ sv)}{s}\Big\|_Y$$

$$\leq \operatorname{Lip}(f) \lim_{n\to\infty} \|v_n - v\|_X \ = \ 0.$$

By hypothesis (iii), fixing $n \geq 1$, we have that $\lim_{s\to 0^+} \phi_n(s) \ = \ A(\cdot, v_n)$, uniformly in $x \in X$. Consequently

$$A(\cdot, v_n) \in \mathcal{C}_b(X,Y), \quad n \geq 1.$$

Hence there exists the following limit in $\mathcal{C}_b(X,Y)$,

$$\lim_{s\to 0^+} \phi(s) \ = \ Df(\cdot)(v) \ = \lim_{n\to\infty} A(\cdot, v_n). \tag{2.2.3}$$

We have just proved that for any $v \in X$, $Df(\cdot)(v) \in \mathcal{C}_b(X,Y)$. Thus the proof is complete. ■

We present our first density result. The proof uses as tool the heat semigroup on abstract Wiener spaces similarly to Goodman [37].

Theorem 2.2.2 *Let X be a real separable Banach space. Then $\mathcal{C}^1_s(X)$ is dense in $\mathcal{C}_b(X)$.*

Proof We shall use the fact that $\mathcal{C}^{0,1}_b(X)$ is dense in $\mathcal{C}_b(X)$ (see for instance Gross [41, §3.2.1]).

Moreover we regard X as an abstract Wiener space. Indeed there exists a separable Hilbert space H_0 and a non degenerate symmetric Gaussian measure p_1 such that (X, H_0, p_1) is an abstract Wiener space (see §1.1.1).

Let O_t be the heat semigroup on $\mathcal{C}_b(X)$ (see (2.1.3)), we show that

$$O_t\left(\mathcal{C}^{0,1}_b(X)\right) \subset \mathcal{C}^1_s(X), \qquad t > 0. \tag{2.2.4}$$

Once (2.2.4) is proved, the assertion follows. Indeed for any $g \in \mathcal{C}_b(X)$, for any $\epsilon > 0$, there exists $l \in \mathcal{C}^{0,1}_b(X)$ such that $\|g - l\|_0 \leq \epsilon$. Now the inequality

$$\|g - \ O_t l\|_0 \ \leq \ \|g \ - l\|_0 \ + \ \|l - O_t l\|_0 \tag{2.2.5}$$

allows us to conclude, using that O_t is strongly continuous.

Fix $f \in \mathcal{C}^{0,1}_b(X)$ and $t > 0$. We shall apply Lemma 2.2.1, using the density of H_0 in X. Thanks to the Cameron-Martin formula (1.1.9) (see Gross [41, §9] for more details) we obtain, for any $h \in H_0$, $x \in X$,

$$\begin{aligned}
&\frac{O_t f(x+sh) - \ O_t f(x)}{s} \\
&= \frac{1}{s}\Big(\int_X f(x+y)p_t(sh,dy) \ - \int_X f(x+y)p_t(dy)\Big) \\
&= \ \frac{1}{s}\int_X f(x+y)\Big[\exp\Big(-\frac{s^2}{2t}\|h\|^2_{H_0} + \ \frac{s}{t}R_h(y)\Big) - 1\Big]p_t(dy),
\end{aligned} \tag{2.2.6}$$

where we have set $R_h = R_h^t$, $h \in H_0$, for convenience. Moreover, by the Mean Value Theorem, we find the following estimate, for any $h \in H_0$,

$$\Big|\frac{1}{s}\Big(\exp\big[-\frac{1}{2t}s^2\|h\|_{H_0}^2 + \frac{s}{t}R_h(y)\big] - 1\Big)\Big|$$

$$\leq \Big(\frac{\|h\|_{H_0}^2 + |R_h(y)|}{t}\Big)\ \exp\Big[\frac{1}{t}\,|R_h(y)|\Big],\quad 0 < s < 1,\ y \in X.$$

Notice that $\exp[\frac{1}{t}\,|R_h(\cdot)|]$ is p_t-integrable on X, since R_h is a Gaussian random variable. Letting $s \to 0^+$ in the last term of (2.2.6), by the Dominated Convergence Theorem, we infer

$$\lim_{s\to 0^+} \frac{O_t f(x+sh) - O_t f(x)}{s} = \frac{1}{t}\int_X f(x+y)\,R_h(y)\,p_t(dy),\quad h \in H_0,\ x \in X. \quad (2.2.7)$$

Of course formula (2.2.7) holds as well when f is only a Borel bounded function.

Now we prove that in addition, for any $h \in H_0$, the limit in (2.2.7) is uniform in $x \in X$. There results

$$\lim_{s\to 0^+}\sup_{x\in X}\Big|\frac{O_t f(x+sh) - O_t f(x)}{s} - \frac{1}{t}\int_X f(x+y)R_h(y)p_t(dy)\Big|$$

$$\leq \|f\|_0 \lim_{s\to 0^+}\int_X \Big|\frac{1}{s}\Big[\exp\Big(-\frac{s^2}{2t}\|h\|_{H_0}^2 + \frac{s}{t}R_h(y)\Big) - 1\Big] - \frac{1}{t}R_h(y)\Big|\,p_t(dy) = 0, \quad (2.2.8)$$

by the Dominated Convergence Theorem.

Notice that $f \in \mathcal{C}_b^{0,1}(X)$ implies that $O_t f \in \mathcal{C}_b^{0,1}(X)$, $t > 0$, thanks to the estimate

$$|O_t f(x) - O_t f(z)| \leq \int_X |f(x+y) - f(z+y)|p_t(dy) \leq \mathrm{Lip}\ (f)|x-z|, \quad (2.2.9)$$

for any $x,\ z \in X$. By Lemma 2.2.1 and by (2.2.8), we get that $O_t f$ is Hadamard differentiable on X, with the Hadamard derivative $DO_t f \in \mathcal{C}_s(X, X')$, $t > 0$. Thus $O_t f \in \mathcal{C}_s^1(X)$, $t > 0$, and the proof is complete. ∎

Remark 2.2.3 Let $f \in \mathcal{C}_b^{0,1}(X)$, by the above proof, applying formula (2.2.3), we infer an explicit formula for the Hadamard derivative $DO_t f$, $t > 0$.
Indeed for any $u \in X$, take any sequence $(h_n) \subset H_0$, such that $h_n \to u$ as $n \to \infty$. This way we find

$$DO_t f(x)(u) = \lim_{n\to\infty}\frac{1}{t}\int_X f(x+y)\ R_{h_n}(y)\ p_t(dy),\quad x,\,u \in X,\ t > 0. \quad (2.2.10)$$

and the limit is uniform in $x \in X$. Moreover notice that it holds

$$\|DO_t f\|_0 \leq \mathrm{Lip}\ (O_t f) \leq \mathrm{Lip(f)}\ ,\quad t > 0. \quad (2.2.11)$$ ∎

Remark 2.2.4 Goodman actually proved that for any separable Banach space X, the space $Q^1(X)$ is dense in $\mathcal{C}_b(X)$ (see [37]).

To introduce the space $Q^1(X)$, he used the notion of "quasi-differentiability" that we briefly recall. A function $f : X \to \mathbb{R}$ is said to be quasi-differentiable at $x \in X$, if there exists $\eta_x \in X'$ such that for each function g from a neighbourhood of 0 in $\mathbb{R}$ into X, which is differentiable at 0 and takes value x at 0, the function $f \circ g$ has a derivative at 0 equal to $\eta_x(g'(0))$. If f is quasi-differentiable at $x \in X$, the functional η_x is said to be the quasi-derivative of f at x.

$Q^1(X)$ is the space of all functions f in $\mathcal{C}_b(X)$, that are quasi-differentiable at each point of X, having a bounded quasi-derivative Df such that

$$Df(\cdot)(\cdot) \ : X \times X \ \to \ \mathbb{R} \ \text{ is continuous.}$$

At present it is known (see Flett [34](§4.2.8)) that quasi-differentiability is equivalent to Hadamard differentiability. Further, invoking (2.1.2) of Lemma 2.1.1, we can state that $\mathcal{C}_s^1(X) \subset Q^1(X)$.

Moreover the inclusion is strict even when $X = \mathbb{R}$. Indeed one verifies readily that
$Q^1(\mathbb{R}) = \{f \in \mathcal{C}_b(\mathbb{R})$, differentiable, having a continuous and bounded derivative $\}$,
$\mathcal{C}_s^1(\mathbb{R}) = \{f \in \mathcal{C}_b(\mathbb{R})$, differentiable, having a uniformly continuous and bounded derivative $\}$.

Take for instance $g(x) = x^{-1} \sin(x^2)$, $x \in \mathbb{R}$. We have that $g \in Q^1(\mathbb{R})$ but $g \notin \mathcal{C}_s^1(\mathbb{R})$. ∎

We point out that heat semigroup on $\mathcal{C}_b(X)$ does not help us to approximate uniformly any function $f \in \mathcal{C}_b(X)$ by means of mappings which belong to $\mathcal{C}_b^{0,1}(X)$. This is stated in the next result.

Proposition 2.2.5 *Let O_t be the heat semigroup on $\mathcal{C}_b(X)$, where X is a real separable Banach space. Then, for any $t > 0$, it holds*

$$O_t(\,\mathcal{C}_b(X)\,) \not\subset \mathcal{C}_b^{0,1}(X).$$

Proof Denote by $\mathcal{A}$ the infinitesimal generator of O_t. Gross has proved (see [41, Theorem 3]) that for any $g \in \mathcal{C}_b^{0,1}(X)$, there results $O_t g \in \mathrm{Dom}(\mathcal{A})$, for all $t > 0$.

Assume, by contradiction, that there exists $t_0 > 0$ such that $O_{t_0}(\mathcal{C}_b(X)) \subset \mathcal{C}_b^{0,1}(X)$. It follows that, for any $\epsilon > 0$,

$$O_{t_0+\epsilon} f \ = \ O_\epsilon \, O_{t_0} f \ \in \mathrm{Dom}(\mathcal{A}), \quad f \in \mathcal{C}_b(X).$$

But this is not true, since it has been recently proved in Guiotto [42] that O_t is not *eventually differentiable* ([1]), see also Desch and Rhandi [27], Van Neerven and Zabczyk [82]. ∎

[1] Let P_t be a strongly continuous linear semigroup on a Banach space X. P_t is said to be *eventually differentiable* if there exists $\hat{t} \geq 0$ such that for any $x \in X$, the map $t \mapsto P_t x$, from $(\hat{t}, \infty)$ into X is differentiable.

Let $X = (X, H_0, p_1)$ be an abstract Wiener space and Y be an arbitrary real Banach space. In order to prove other density results, let us note that the heat semigroup can be also defined in $\mathcal{C}_b(X,Y)$. We denote by $\hat{O}_t$ such a semigroup, defined as follows

$$\hat{O}_t f(x) \stackrel{\text{def}}{=} \int_X f(x+y) p_t(dy), \quad f \in \mathcal{C}_b(X,Y), \ \ t>0, \ \ x \in X, \tag{2.2.12}$$

where the integral is in Bochner's sense. Indeed X is separable and so, for any $f \in \mathcal{C}_b(X,Y)$, the continuity of f implies that the range of f is separable in Y.

Gross has proved that O_t is strongly continuous on $\mathcal{C}_b(X)$ (see Proposition 6 of Gross [41]). Following his proof, it is possible to prove, without difficulties, that also $\hat{O}_t$ is a strongly continuous semigroup on $\mathcal{C}_b(X,Y)$.

We present our second density result, concerning Hilbert spaces.

Theorem 2.2.6 *Let H , K be real Hilbert spaces and assume that H is separable. Then $\mathcal{C}^1_s(H,K)$ is dense in $\mathcal{C}_b(H,K)$.*

Proof We shall use the fact that $\mathcal{C}^{0,1}_b(H,K)$ is dense in $\mathcal{C}_b(H,K)$ (see Valentine [80] and Tsar'kov [79]).

Moreover we consider H as an abstract Wiener space (H, H_0, p_1). We recall that $p_1 = \mathcal{N}(0,Q)$, where Q is a symmetric positive trace class operator on H and $H_0 = Q^{1/2}H$ (see §1.1.2). We denote by $\hat{O}_t$ the heat semigroup in $\mathcal{C}_b(H,K)$, defined by (2.2.12). We argue as in the proof of Theorem 2.2.2.

Notice that any map $f \in \mathcal{C}_b(H,K)$ can be pointwise approximated by a sequence of simple functions (f_n) such that $\|f_n(x) - f(x)\|_K \downarrow 0$, as $n \to \infty$, for any $x \in H$ (see Lemma 1.1 of Da Prato and Zabczyk [23]). Using this fact and the Cameron-Martin formula, we obtain readily, for any $f \in \mathcal{C}_b(H,K)$, $t>0$,

$$\hat{O}_t f(x+h) \ = \ \int_H f(x+y) \ \exp\Big(- \frac{1}{2t}\|h\|^2_H + \frac{1}{t} R_h(y) \Big) p_t(dy), \quad x \in H, \quad h \in H_0.$$

Therefore formula (2.2.8) also holds if O_t is replaced with $\hat{O}_t$ and $f \in \mathcal{C}_b(H,K)$. Applying Lemma 2.2.1 we find that

$$\hat{O}_t(\mathcal{C}^{0,1}_b(H,K)) \ \subset \ \mathcal{C}^1_s(H,K), \qquad t>0. \tag{2.2.13}$$

Arguing as for formula (2.2.5) we obtain the assertion. ∎

It is known that for any Hilbert space H, $\mathcal{C}^{1,1}_b(H)$ is dense in $\mathcal{C}_b(H)$ (see Lasrly and P. L. Lions [52]). We use this result to prove the following theorem.

Theorem 2.2.7 *Let H, be a real separable Hilbert space, with inner product $< \cdot, \cdot >$ and norm $| \cdot |$. Then $\mathcal{C}^2_s(H)$ is dense in $\mathcal{C}_b(H)$.*

Proof We consider H as an abstract Wiener space (H, H_0, p_1). Let us indentify H with H'. Consider the following two heat semigroups:

$$O_t \ \text{on} \ \ \mathcal{C}_b(H) \ \ \text{and} \ \ \hat{O}_t \ \text{on} \ \ \mathcal{C}_b(H,H), \tag{2.2.14}$$

both defined by integrals with respect to p_t (see (2.2.12)). Arguing as in the proof of Theorem 2.2.2, see formula (2.2.5), in order to prove our assertion it is enough to show that for any $f \in \mathcal{C}_b^{1,1}(H)$ and $t > 0$, we have $O_t f \in \mathcal{C}_s^2(H)$.

To this end fix $f \in \mathcal{C}_b^{1,1}(H)$ and $t > 0$. First we prove that $O_t f$ is Gâteaux differentiable on H. Denote by Df the Fréchet derivative of f. Let $x \in H$, for any $v \in H$, $s \in (0,1]$ we have

$$\lim_{s \to 0^+} \frac{O_t f(x+sv) - O_t f(x)}{s} = \int_H < Df(x+y), v > \ p_t(dy),$$

since f is a Lipschitz continuous map and so we can pass to the limit under the integral sign, by the Dominated Convergence Theorem. This way we obtain that there exists the Gâteaux derivative: $D_G O_t f(x)$ at any $x \in H$ and further that it holds, for every $v \in H$,

$$\begin{aligned} &< D_G O_t f(x), v > = O_t(< Df(\cdot), v >)\,(x) \\ &= < \int_H Df(x+y)\, p_t(dy),\, v > = < \hat{O}_t(Df)(x), v >, \end{aligned} \tag{2.2.15}$$

$$\text{so that} \quad D_G O_t f(x) = \hat{O}_t Df(x), \quad x \in H.$$

Now $Df \in \mathcal{C}_b(H,H)$ implies that $\hat{O}_t Df \in \mathcal{C}_b(H,H)$, with $\omega_{\hat{O}_t Df} \le \omega_{Df}$. Indeed, for any $t > 0$,

$$\int_H |Df(x+y) - Df(z+y)|\, p_t(dy) \le \omega_{Df}(|x-z|), \ \ x, z \in H.$$

Invoking a well known result about differentiability we deduce that $O_t f$ is also Fréchet differentiable on H and so (2.2.15) holds with the Gâteaux derivative replaced by the Fréchet derivative.

Now remark that, by the assumptions, $Df \in \mathcal{C}_b^{0,1}(H,H)$. Applying formula (2.2.13) in the proof of Theorem 2.2.6 (with $K = H$) we obtain that $\hat{O}_t Df \in \mathcal{C}_s^1(H,H)$. This is equivalent, since $DO_t f = \hat{O}_t Df$, to say that there exists the second Hadamard derivative of $O_t f$ and that it belongs to $\mathcal{C}_s(H, \mathcal{L}_s(H))$. Thus the proof is complete. ■

Consider the following family of non linear and continuous operators L_t on $\mathcal{C}_b(H)$, $t > 0$, defined as follows

$$L_t f(x) = \sup_{z \in H} \Big\{ \inf_{y \in H} \Big[f(y) + \frac{|z-y|^2}{2t} \Big] - \frac{|z-x|^2}{t} \Big\}, \tag{2.2.16}$$

where $f \in \mathcal{C}_b(H)$ and $x \in H$. These operators are related to viscosity solutions of Hamilton-Jacobi equations (see Crandall and P.L. Lions [16]). In Lasrly and Lions [52] it is shown that, for any $f \in \mathcal{C}_b(H)$, it holds

$$L_t f \in \mathcal{C}_b^{1,1}(H), \ \ t > 0, \ \ \text{and} \quad \lim_{t \to 0^+} \|L_t f - f\|_0 = 0. \tag{2.2.17}$$

Using this fact and Theorem 2.2.7, we can obtain the following useful result.

Corollary 2.2.8 *Let L_t be the operators defined in (2.2.16) and O_t be the heat semigroup on $\mathcal{C}_b(H)$, where H is a real separable Hilbert space. For any $f \in \mathcal{C}_b(H)$, one has*

$$O_t L_t f \in \mathcal{C}^2_s(H), \ \ t > 0, \ \ \text{and} \quad \lim_{t \to 0^+} \|O_t L_t f - f\|_0 = 0.$$

Proof By the proof of Theorem 2.2.7, we already know that $O_t L_t f \in \mathcal{C}^2_s(H)$, $t > 0$. Moreover we have

$$\begin{aligned} |O_t L_t f(x) - f(x)| &\leq |O_t L_t f(x) - O_t f(x)| + |O_t f(x) - f(x)| \\ &\leq \int_H |L_t f(x+y) - f(x+y)| p_t(dy) + \|O_t f - f\|_0 \\ &\leq \|L_t f - f\|_0 + \|O_t f - f\|_0, \quad x \in H, \ t > 0. \end{aligned}$$

Letting $t \to 0^+$ in the last term, we find 0, by using (2.2.17) and the fact the O_t is strongly continuous. This completes the proof. ∎

Remark 2.2.9 We want to show how the heat semigroup can be used to improve other approximation results.
In Konyagin and Tsar'kov [48] it is mentioned, without proof, the following Tsar'kov result:

let H, K be real Hilbert spaces; then any uniformly continuous map $g : H \to K$ is uniformly approximable by Fréchet differentiable maps $f : H \to K$, having a bounded derivative.

We define the space $\mathcal{C}^1_{s,F}(H, K) = \{f \in \mathcal{C}^1_s(H, K)$ which are Fréchet differentiable in $H\}$. Invoking the Tsar'kov theorem we can prove:

let H, K be real Hilbert space and assume that H be separable, then $\mathcal{C}^1_{s,F}(H, K)$ is dense in $\mathcal{C}_b(H, K)$.

Let $\hat{O}_t$ be the heat semigroup in $\mathcal{C}_b(H, K)$ (see the proof of Theorem 2.2.6). Fix $g \in \mathcal{C}_b(H, K)$. For any $\epsilon > 0$, by the above Tsar'kov theorem, we can choose a function $f \in \mathcal{C}^{0,1}_b(H, K)$ which has a bounded Fréchet derivative on H and such that $\|g - f\|_0 \leq \epsilon$.

Using the inequality $\|g - \hat{O}_t f\|_0 \leq \|g - f\|_0 + \|f - \hat{O}_t f\|_0$, $t > 0$, in order to get the assertion it is enough to show that $\hat{O}_t f \in \mathcal{C}^1_{s,F}(H, K)$ for any $t > 0$.
Fix $t > 0$, by formula (2.2.13) we know that $\hat{O}_t f \in \mathcal{C}^1_s(H, K)$. Thus we only prove that $\hat{O}_t f$ is Fréchet differentiable on H and it holds:

$$D\hat{O}_t f(x)(v) = \hat{O}_t[Df(\cdot)(v)](x), \quad x, \ v \in H, \tag{2.2.18}$$

where $D\hat{O}_t f$ and Df are Fréchet derivatives. To establish (2.2.18) we can not argue as in (2.2.15), since Df is not supposed to be continuous. However fix $x \in H$ and let C be the unit closed ball of H. The assertion (2.2.18) is equivalent to prove that

$$\lim_{s \to 0^+} \sup_{v \in C} \Big| \frac{\hat{O}_t f(x + sv) - \hat{O}_t f(x)}{s} - \int_H Df(x+y)(v) p_t(dy) \Big|_K = 0.$$

We define $\Theta : (0, 1] \times C \times H \to \mathbb{R}$, for any $s \in (0, 1]$, $v \in C$, $y \in H$,

$$\Theta(s,v,y) \;=\; \|\frac{f(x+y+sv)-f(x+y)}{s} \;-\; Df(x+y)(v)\|_K.$$

This way in order to get (2.2.18), it is sufficient to verify that

$$\lim_{s\to 0^+}\sup_{v\in C}\int_H \Theta(s,v,y)\,p_t(dy) = \;0.$$

To this purpose, take a countable dense set D in C. Then since $\Theta(s,\cdot,y)$ is uniformly continuous for any $s \in (0,1],\ y \in H$, we have

$$\sup_{v\in D}|\Theta(s,v,y)| \;=\; \sup_{v\in C}|\Theta(s,v,y)|, \quad s\in(0,1],\ y\in H.$$

Now for any $s \in (0,1]$, $v \in C$, $\Theta(s,v,\cdot)$ is a Borel function and so $\sup_{v\in D}|\Theta(s,v,y)|$ is still Borel, since D is countable.

Moreover we have $|\Theta(s,v,y)| \leq\; 2\|Df\|_0,\;\; s\in(0,1],\ v\in C,\ y\in H.$ By the inequality

$$\sup_{v\in C}\int_H |\Theta(s,v,y)|p_t(dy) \;\leq\; \int_H \sup_{v\in D}|\Theta(s,v,y)|\,p_t(dy),$$

as $s \to 0^+$ in the right-hand side, we get 0, applying the Dominated Convergence Theorem. Thus (2.2.18) is proved. ∎

Thanks to the previous density theorems we can also approximate uniformly mappings which are defined on a subset S of a separable Banach space X. To this end we need the following McShane result (see [57]).

Let (M,d) be a metric space with metric d and let A be a subset of M. Then any map $f:\ A\to\mathbb{R}$, uniformly continuous and bounded can be extended to a uniformly continuous and bounded map $\hat f : M\to\mathbb{R}$ (i.e. $\hat f(x) = f(x),\ \ x\in A$),

$$\hat f(x) \;=\; \sup_{z\in A}\;[f(z)\;-\omega_f(d(x,z))],\;\; x\in M, \tag{2.2.19}$$

where ω_f denotes the modulus of continuity of f. Moreover $\hat f$ has the same bounds and the same modulus of continuity of f.

As a direct consequence of the McShane theorem and of Theorems 2.2.2 and 2.2.7, we find

Theorem 2.2.10 *Let X be a real separable Banach space and let S be a subset of X. The following statements hold:*

(i) the restrictions to S of functions which belong to $\mathcal{C}^1_s(X)$ are dense in $\mathcal{C}_b(S)$;

(ii) if X is a Hilbert space, then the restrictions to S of functions which belong to $\mathcal{C}^2_s(X)$ are dense in $\mathcal{C}_b(S)$.

Remark 2.2.11 Let Ω be an open set of a separable Hilbert space H. By Theorem 2.2.10, we know that $\mathcal{C}_s^2(\Omega)$ is dense in $\mathcal{C}_b(\Omega)$. However we point out that $\mathcal{C}_b^2(\Omega)$ ([2]) is not dense in $\mathcal{C}_b(\Omega)$.

We discuss this unexpected fact that is a consequence of a Nemirovskii and Semenov's result (see [59]).

Nemirovskii and Semenov have constructed a map $f_0 \in \mathcal{C}_b(B_0)$, where $B_0 = B(0,1)$ denotes the unit open ball of H, that is not uniformly approximable by maps which belong to $\mathcal{C}_b^2(B)$.

Choose $x \in \Omega$ and take an open ball $B = B(x,r) \subset \Omega$. Define a map f on B as follows: $f(z) = f_0(\frac{z-x}{r}), \quad z \in B$. Of course we have that $f \in \mathcal{C}_b(B)$. By the McShane extension theorem (see (2.2.19)), we can extend f to a map $\hat{f} \in \mathcal{C}_b(\Omega)$.

Now assume by contradiction that $\mathcal{C}_b^2(\Omega)$ is dense in $\mathcal{C}_b(\Omega)$. Then there exists a sequence $(\hat{f}_n) \subset \mathcal{C}_b^2(\Omega)$ such that $\|\hat{f}_n - \hat{f}\|_{0,\Omega} \to 0$ as $n \to \infty$. Denote by f_n, the restrictions of $\hat{f}_n$ to B. It follows that (f_n) converges uniformly in B to f.

Now define the maps $g_n(y) = f_n(ry + x), \; y \in B_0$. We have that $g_n \in \mathcal{C}_b^2(\Omega)$ for any $n \geq 1$. Moreover it is easy to show that (g_n) converges uniformly in B_0 to f_0. This contradicts the Nemirovskii and Semenov result and concludes the proof. ∎

2.3 Uniform approximation and Urysohn maps

Here we introduce a connection between uniform approximation of real, uniformly continuous and bounded functions and existence of Urysohn maps.

Let (M,d) be a metric space with metric d and denote by $\mathcal{C}_b(M)$, the Banach space of all real, bounded, uniformly continuous mappings endowed with the sup norm.

Two non empty closed subsets A and B are said to be *separated* if

$$\inf_{x\in A,\, y\in B} d(x,y) > 0.$$

A function $f \in \mathcal{C}_b(M)$ is said to be an *Urysohn function* for the pair (A,B) of separated closed subsets if

$f : M \to [0,1], \quad f(x) = 1$ for any $x \in A, \quad f(y) = 0$ for any $y \in B$.

Remark that for any separated closed subsets A, B in M, the function

$$f_{A,B}(x) = \frac{d(x,B)}{d(x,A) + d(x,B)}, \quad x \in M,$$

is a Lipschitz continuous Urysohn function for (A,B).

It is known (see for instance Piech [64]) that the uniform approximation for functions which belong to $\mathcal{C}_b(X)$ (X is a Banach space), by means of smooth functions, implies the existence of smooth Urysohn functions. Thus by the density results of the previous section follow the next two propositions.

[2] $\mathcal{C}_b^2(\Omega)$ consists of all maps f which are in $\mathcal{C}_b(\Omega)$, having a first and second bounded Fréchet derivative on Ω and such that $D^2 f \in \mathcal{C}_b(\Omega, \mathcal{L}(H))$.

Proposition 2.3.1 *Let H be a separable Hilbert space, for any pair of separated closed subsets A and B, there exists an Urysohn map g for (A, B) that belongs to $\mathcal{C}^2_s(H)$.*

Proof Take a Lipschitz continuous Urysohn map f for (A, B), then by Theorem 2.2.6, there exists a map $h \in \mathcal{C}^2_s(H)$ such that: $h(x) > 3/4$ when $x \in A$ and $h(y) < 1/4$ when $y \in B$. Now consider a function $j \in \mathcal{C}^\infty_b(\mathbb{R})$ defined as follows:

$$j : \mathbb{R} \to [0, 1], \;\; j(s) = 1, \;\; \text{for} \;\; |s| \geq 3/4, \;\; j(s) = 0, \;\; \text{for} \;\; |s| \leq 1/4.$$

Finally set $g = j \circ h$. It turns out that $g \in \mathcal{C}^2_s(H)$ and it is the desired Urysohn map for (A, B). ■

In the same way we can get

Proposition 2.3.2 *Let X be a separable Banach space, for any pair of separated closed subsets A and B, there exists an Urysohn map g for (A, B) that belongs to $\mathcal{C}^1_s(X)$.*

Thus on one hand it is clear the link between uniform approximation and existence Urysohn functions. On the other hand we prove that there is a connection as well. The following theorem shows that the existence of regular Urysohn functions implies the existence of regular uniform approximations.

Theorem 2.3.3 *Let (M, d) be a metric space, with metric d, and $\mathcal{S}(M)$ be a linear subspace of $\mathcal{C}_b(M)$. If for any pair of separated closed subsets A and B there exists a Urysohn function $f \in \mathcal{S}(M)$ for (A, B), then $\mathcal{S}(M)$ is dense in $\mathcal{C}_b(M)$.*

Proof Fix any $m \in M$. There exists $\eta > 0$ such that $C_\eta = \{x \in M,$ such that $d(x, m) \geq \eta\}$ is not empty (otherwise $M = \{m\}$ and the assertion is verified).

Clearly C_η and $\{m\}$ are two separated closed sets. Thus, by hypothesis, there exists a map $\hat{f} \in \mathcal{S}(M)$ such that $\hat{f}(m) = 1$ and $\hat{f}(x) = 0$ if $x \in C_\eta$.

Take any $g \in \mathcal{C}_b(M)$. We can assume that $g(m) = 0$, otherwise we replace g with the map

$$\hat{g} = g - g(m)\hat{f}, \quad \text{where} \quad g(m)\hat{f} \in \mathcal{S}(M).$$

Moreover, since $g = \max(g, 0) - \max(-g, 0)$, in order to prove the thesis it is enough to consider only non negative functions $f \in \mathcal{C}_b(M)$ such that $\inf_{x \in M} f(x) = 0$.

Now we use an inductive argument as in the proof of Lemma 3.2.1 in Gross [41]. Fix $\epsilon > 0$ and let

$$\Lambda_n = \{f \in \mathcal{C}_b(M), \; / \; f(x) \leq n\epsilon, \;\; x \in M\}.$$

We shall show, by induction on n, that for any function in Λ_n there exists a map $h \in \mathcal{S}(M)$ such that $\|f - h\|_0 \leq 2\epsilon$.

The assertion is true if $n = 2$, taking $h = 0$, so we suppose that the assumption is satisfied for all $n \le k$ and prove it for $n = k + 1$, where $k \ge 2$. Let f be in Λ_{k+1} but not in Λ_k and set

$$A = \{x \in M, \ / \ f(x) \ge k\epsilon\}, \quad B = \{x \in M, \ / \ f(x) \le (k-1)\epsilon\}.$$

A, B are two non empty closed subsets. They are also separated thanks to the uniform continuity of f. Hence we can take a map $l \in \mathcal{S}(M)$ that is a Urysohn map for (A, B).

Consider $\epsilon\, l \in \mathcal{S}(M)$. We have that $\epsilon l(x) = \epsilon$ for any $x \in A$ and $\epsilon l(x) = 0$ for any $x \in B$ so that we infer

$$0 \le f(x) - \epsilon l(x) \le k\epsilon, \quad x \in M.$$

It follows that $f - \epsilon l \in \Lambda_k$ and by the induction hypothesis there exists a map $g \in \mathcal{S}(M)$ such that

$$\|f - \epsilon l - g\|_0 \le 2\epsilon.$$

Taking $h = \epsilon l + g \in \mathcal{S}(M)$, we obtain the assertion. The proof is complete. ■

Now using Theorem 2.3.3 and Remark 2.2.11 we deduce the following result, that is proved in Priola [65] and considered in Tessitore and Zabczyk [76].

Corollary 2.3.4 *Let H be a real separable Hilbert space, then there exist two separated closed subsets A and B in H such that they do not admit any Urysohn function which belongs to $\mathcal{C}_b^2(H)$.*

2.4 Uniform approximation of σ-uniformly continuous maps

We review some notions on locally convex topologies. Let $(X, \|\cdot\|_X)$ be a Banach space and let σ be a locally convex Haussdorff topology on X. Γ_σ denotes the family of all seminorms on X which are continuous with respect to σ.

We consider three different spaces of real functions on X.

$\mathcal{C}_\sigma(X) \stackrel{\text{def}}{=} \{f : (X, \sigma) \to \mathbb{R}$, uniformly continuous ([3]) and bounded $\}$.
$\mathcal{C}_\sigma(X)$ turns out to be a Banach space endowed with the sup norm.

$\mathcal{C}_\sigma^{0,1}(X) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_\sigma(X)$, for which there exists $q_f \in \Gamma_\sigma$, a constant $\mathrm{L}(f) > 0$, such that $|f(x) - f(z)| \le \mathrm{L}(f)\, q_f\,(x - z), \quad x, z \in X \}$,

$\mathcal{C}_\sigma^1(X) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_\sigma(X)$, having the Fréchet derivative Df in X such that $Df : (X, \sigma(X, X')) \to X'$ is uniformly continuous and bounded $\}$.

Clearly if σ is weaker than the norm topology of X we have:

[3] A map $f : (X, \sigma) \to \mathbb{R}$ is uniformly continuous if and only if for any $\epsilon > 0$, there exists $\delta > 0$ and $q \in \Gamma_\sigma$ such that for any $x, y \in X$, $q(x - y) \le \delta$ implies that $|f(x) - f(y)| \le \epsilon$

$\mathcal{C}_\sigma(X) \subset \mathcal{C}_b(X)$, $\mathcal{C}_\sigma^{0,1}(X) \subset \mathcal{C}_b^{0,1}(X)$, $\mathcal{C}_\sigma^1(X) \subset \mathcal{C}_b^1(X)$ and the inclusions are strict. The following lemma shows that $\mathcal{C}_\sigma^{0,1}(X)$ is dense in $\mathcal{C}_\sigma(X)$. It is a straightforward variation of Lemma 3.2.1 in Gross [41], we state it without proof.

Lemma 2.4.1 *Let (V, σ) be a real locally convex Haussdorff space. Then $\mathcal{C}_\sigma^{0,1}(V)$ is dense in $\mathcal{C}_\sigma(V)$.*

Now we are ready to prove the following result.

Theorem 2.4.2 *Let X be a real separable Banach space, with the unit closed ball denoted by C, and σ be a locally convex Haussdorff topology on X such that:*
(i) σ is weaker than the norm topology;
(ii) (C, σ), i.e. C endowed with σ, is compact.
Then $\mathcal{C}_\sigma^1(X)$ is dense in $\mathcal{C}_\sigma(X)$.

Proof As in the proof of Theorem 2.2.2, we consider X has an abstract Wiener space (X, H_0, p_1) and denote by O_t the heat semigroup on $\mathcal{C}_b(X)$. It follows, by easy computations, that if $g \in \mathcal{C}_\sigma(X)$ then $O_t g \in \mathcal{C}_\sigma(X)$ for any $t > 0$.

By Lemma 2.4.1, arguing as for formula (2.2.5), to prove the assertion it is enough to verify that for any $f \in \mathcal{C}_\sigma^{0,1}(X)$ then $O_t f \in \mathcal{C}_\sigma^1(X)$, for any $t > 0$. Thus fix $f \in \mathcal{C}_\sigma^{0,1}(X)$ and $t > 0$.

First we remark that, by Hypothesis (i), $f \in \mathcal{C}_b^{0,1}(X)$ and so, by formula (2.2.4), $O_t f \in \mathcal{C}_s^1(X)$. We denote by $DO_t f$ the Hadamard derivative of $O_t f$.

Let $q \in \Gamma_\sigma$ such that

$$|f(x) - f(y)| \ \leq \ \mathrm{L}(f)\, q(x-y), \quad x, y \in X.$$

We get easily

$$|O_t f(x) - O_t f(y)| \ \leq \ \mathrm{L}(f)\, q(x-y), \qquad x, y \in X.$$

Define the maps:

$$\phi_s \ : \ (C, \sigma) \ \to \ \mathcal{C}_\sigma(X), \quad s \in (0,1] \quad \text{such that:}$$

$$\phi_s(v) \ = \ \frac{O_t f(\cdot + sv) - O_t f(\cdot)}{s}, \qquad s \in (0,1],\ v \in C. \tag{2.4.1}$$

It is possible to prove, taking into account formulas (2.2.10) and (2.2.3), that for any $v \in C$,

$$\lim_{s \to 0^+} \ \frac{O_t f(x + sv) - O_t f(x)}{s} \ = \ DO_t f(x)(v)$$

and this limit is uniform in $x \in X$. Consequently, for any $v \in C$,

$$\lim_{s \to 0^+} \ \phi_s(v) \ = \ DO_t f(\cdot)(v) \quad \text{in } \mathcal{C}_\sigma(X). \tag{2.4.2}$$

Take any sequence $s_n \subset (0,1]$, such that $s_n \to 0$. By formula (2.4.2), for any $v \in C$, the sequence $(\phi_{s_n}(v))$ is relatively compact in $\mathcal{C}_\sigma(X)$.

Further (ϕ_{s_n}) is an equicontinuous sequence of maps in $\mathcal{C}_\sigma(C, \mathcal{C}_\sigma(X))$([4]), since it holds:

$$\|\phi_{s_n}(v) - \phi_{s_n}(v')\|_{\mathcal{C}_\sigma(X)} \;\le\; \mathrm{L}(f)\; q(v - v'),\;\; v, v' \in C.$$

Therefore applying the Arzela - Ascoli Theorem (as in the proof of Lemma 2.2.1) we deduce that $\lim_{n\to\infty} \phi_{s_n}(v) = DO_t f(\cdot)(v)$, uniformly in $v \in C$. Consequently for the arbitrariness of (s_n), we deduce that

$$\lim_{s\to 0^+} \sup_{v\in C} \|\phi_s(v) - DO_t f(\cdot)(v)\|_{\mathcal{C}_\sigma(X)}$$

$$= \lim_{s\to 0^+} \sup_{v\in C} \sup_{x\in X} \Big| \frac{O_t f(x+sv) - O_t f(x)}{s} - DO_t f(x)(v) \Big| = 0. \tag{2.4.3}$$

In particular this formula entails that $O_t f$ is Fréchet differentiable in X. Moreover we also obtain that

$$DO_t f(\cdot)(\cdot) : (C, \sigma) \times (X, \sigma) \to \mathbb{R} \text{ is uniformly continuous and bounded.}$$

This fact yields that $DO_t f$ is uniformly continuous and bounded from (X, σ) into X'; indeed we have:

$$\sup_{v\in C} |DO_t f(x)(v) - DO_t f(z)(v)| \;=\; \|DO_t f(x) - DO_t f(z)\|_{X'},\;\; x, z \in X.$$

The proof is complete. ∎

Let X be a separable reflexive Banach space. On X we consider the weak topology $\sigma(X, X')$. It is well known that the unit closed ball C in X, is compact with respect to $\sigma(X, X')$. Thus by the above theorem we deduce the following result.

Corollary 2.4.3 *Let $(X, \|\cdot\|_X)$ be a real separable reflexive Banach space and consider the weak topology $\sigma = \sigma(X, X')$ on X. Then $\mathcal{C}^1_\sigma(X)$ is dense in $\mathcal{C}_\sigma(X)$.*

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[4] $\mathcal{C}_\sigma(C, \mathcal{C}_\sigma(X))$ denotes the Banach space of all maps $g : (C, \sigma) \to \mathcal{C}_\sigma(X)$ which are uniformly continuous and bounded, endowed with the usual sup norm.

Chapter 3

Some results on the heat semigroup O_t

3.1 Introduction

In this chapter we are concerned with the heat semigroup O_t in $\mathcal{C}_b(H)$ (see (2.1.3)), where H is a real separable Hilbert space. O_t is a semigroup of bounded linear operators in $\mathcal{C}_b(H)$, associated with the following Cauchy problem:

$$\begin{cases} \partial_t u(t,x) \; = \; \frac{1}{2}\mathrm{Tr}\,[QD^2u(t,x)], \;\; t \geq 0, \;\; x \in H, \\ u(0,x) = \; f(x), \qquad x \in H, \end{cases} \tag{3.1.1}$$

where Q is a self-adjoint positive trace operator on H and $f \in \mathcal{C}_b(H)$. This means that for any f "regular" enough, $u(t,x) = O_t f(x)$ solves (3.1.1).
The material is organized as follows.

In Section 3.2 we review some important results on the heat semigroup available in the literature, providing self-contained proofs. These results clarify the fact that O_t has very different properties according to H is finite or infinite dimensional.
In particular Theorem 3.2.1, proved in Zabczyk and Van Neerven [82], shows that O_t is not eventually norm-continuous in $\mathcal{C}_b(H)$ and consequently not analytic if H is infinite dimensional.

We also study regularizing effects of O_t. On this subject our main reference is the foundamental Gross paper [41] that treats the problem in the more general setting of abstract Wiener spaces. In Proposition 3.2.2 we show that for any $f \in \mathcal{C}_b(H)$, it turns out that $O_t f \in \mathcal{C}_Q^\infty(H)$. This result follows by Proposition 9 of Gross [41], see also Elson [32], Kuo [50] and Piech [64].

A typical regularity problem arising in infinite dimensions is whether the bounded linear operator $D_Q^2 O_t f(x)$, $x \in H$, $t > 0$, $f \in \mathcal{C}_b(H)$, is compact, or of Hilbert-Schmidt type, or of trace class, etc.. This problem is of interest in view of the treatment of second order elliptic PDE's with variable coefficients, involving infinitely many variables (see Part II). On this subject Gross has shown that $D_Q^2 O_t f(x)$ is of Hilbert-Schmidt type and further $D_Q^2 O_t f \in \mathcal{C}_b(H, \mathcal{L}_2(H))$, $t > 0$, $f \in \mathcal{C}_b(H)$. In Proposition 3.2.3, we will provide a simpler and direct proof of this fact.

Section 3.3 contains the main theorem of this chapter. This is concerned with the (infinitesimal) generator $\mathcal{A}$ of the heat semigroup. In order to present this result, we recall that when $H = \mathbb{R}^n$, it is well known (see for instance Lunardi [55]) that

$$D(\mathcal{A}) = \{ g \in \bigcap_{p \geq 1} W^{2,p}_{loc}(\mathbb{R}^n) : \ \ g, \ \sum_{k=1}^{n} \lambda_k D_{kk} g \in \mathcal{C}_b(\mathbb{R}^n) \}$$

and $\mathcal{A}g = \frac{1}{2} \sum_{k=1}^{n} \lambda_k D_{kk} g, \ \ g \in D(\mathcal{A})$. However, when H is infinite dimensional, a similar characterization of $\mathcal{A}$ is not known. In Theorem 3.3.2, proved in Priola [67] and [68], we give a new contribution to this problem and extend a classical theorem due to Gross (see Theorem 3 and Corollary 3.2 of Gross [41]). We consider the following linear operator $\mathcal{A}_1$,

$$\begin{cases} D(\mathcal{A}_1) = \ \{ f \in \mathcal{C}^2_Q(H) \text{ such that } D^2_Q f(x) \in \mathcal{L}_1(H), \ \ x \in H, \text{ and} \\ D^2_Q f \in \mathcal{C}_b(H, \mathcal{L}_1(H)) \}; \\ \\ \mathcal{A}_1 : D(\mathcal{A}_1) \ \rightarrow \ \mathcal{C}_b(H), \ \ \mathcal{A}_1 f(x) \stackrel{\text{def}}{=} \frac{1}{2} \text{Tr} \, [D^2_Q f(x)], \ f \in D(\mathcal{A}_1), \ x \in H, \end{cases}$$

and prove that $\mathcal{A}$ extends $\mathcal{A}_1$ (i.e. for any $f \in D(\mathcal{A}_1)$, it turns out that $f \in D(\mathcal{A})$ and further $\mathcal{A}f = \mathcal{A}_1 f$). From this fact it will also follow that $\mathcal{A}$ is the closure of $\mathcal{A}_1$ or equivalently that $D(\mathcal{A}_1)$ is a **core** ([1]) for $\mathcal{A}$.

In §3.3 we also prove Theorem 3.3.5, that generalizes Theorem 4.1 in Cannarsa and Da Prato [12], introducing a very small core for $\mathcal{A}$. This core consists of functions, having the second derivative in the Hadamard sense, see Chapter 1 and Theorem 2.2.7.

Finally the last section is devoted to study the interpolation spaces $\mathcal{D}_{\mathcal{A}}(\theta, \infty) = (\mathcal{C}_b(H), D(\mathcal{A}))_{\theta,\infty}$. When $H = \mathbb{R}^n$, it is well known that it holds: $(C_b(\mathbb{R}^n), D(\mathcal{A}))_{\theta/2,\infty} = C^\theta_b(\mathbb{R}^n)$, for $\theta \in]0, 1[$. On the contrary, in infinite dimensions, only the continuous embedding

$$(C_b(H), D(\mathcal{A}))_{\theta/2,\infty} \subset C^\theta_Q(H), \tag{3.1.2}$$

has been proved, see Cannarsa and Da Prato [12]. The characterization of $\mathcal{D}_{\mathcal{A}}(\theta, \infty)$ is not known. In this direction, see Theorem 3.4.3, we show that the inclusion in (3.1.2) is *strict*. This result, proved in Priola and Zambotti [70], will be also discussed in the next chapter in connection with second order elliptic equations. The proof uses a recent result of Van Neerven and Zabczyk, about the norm discontinuity of the heat semigroup, previously mentioned.

3.2 Regularity properties for O_t

Let H be a real separable Hilbert space with inner product $< \cdot, \cdot >$ and norm $| \cdot |$. We recall that $\mathcal{L}_1(H)$ denotes the subspace of $\mathcal{L}(H)$ ([2]) of all trace class operators (see

[1] Let $\mathcal{B} : D(\mathcal{B}) \subset X \rightarrow X$ be a closed operator on a Banach space X. A subspace Y of $D(\mathcal{B})$ is said to be a *core* for $\mathcal{B}$ if Y is dense in $D(\mathcal{B})$ with respect to the graph norm: $\|x\|_{D(\mathcal{B})} \stackrel{\text{def}}{=} \|x\|_X + \|\mathcal{B}x\|_X, \ \ x \in D(\mathcal{B})$.

[2] $\mathcal{L}(H)$ stands for the Banach space of all bounded linear operators on H, endowed with the norm: $\|T\|_{\mathcal{L}(H)} = \sup_{|v| \leq 1} |Tv|, \ \ T \in \mathcal{L}(H)$.

Chapter 1 for more details).

Let Q be a positive (i.e. one to one and non negative) self-adjoint trace class operator in H ($\mathrm{Tr}(Q)$ denotes the trace of Q). We fix once and for all an orthonormal basis of H, $\{e_k\}_{k\geq 1}$, that diagonalizes Q:

$$Qx = \sum_{k=1}^{\infty} \lambda_k x_k \, e_k, \quad \text{with} \quad x_k =< x, e_k >, \; \lambda_k > 0, \, x \in H. \tag{3.2.1}$$

Moreover $\mathcal{N}(x, tQ)$ denotes the Gaussian measure on H, with mean $x \in H$ and covariance operator tQ (we refer to §1.1.2 as concerns the main properties of Gaussian measures in Hilbert spaces).

We only recall the basic Cameron-Martin formula in Hilbert spaces. It asserts that the measures $\mathcal{N}(0, tQ)$ and $\mathcal{N}(x, tQ)$, $t > 0$, $x \in H$, are either equivalent or singular. They are equivalent if and only if $x \in Q^{1/2}H$. Further if $x = Q^{1/2}h$, $h \in H$, the Radon-Nikodym derivative of $\mathcal{N}(Q^{1/2}h, tQ)$ with respect to $\mathcal{N}(0, tQ)$, is given, for any $t > 0$, by the following formula:

$$\frac{d\mathcal{N}(Q^{1/2}h, tQ)}{d\mathcal{N}(0, tQ)}(y) = \exp\left[-\frac{1}{2t}|h|^2 + \frac{1}{\sqrt{t}} < (tQ)^{-1/2}y, h >\right], \;\; y \in H, \; \mathcal{N}(0, tQ) - a.e., \tag{3.2.2}$$

where $< (tQ)^{-1/2}\,(\cdot), h >$, see (1.1.17), is a Gaussian random variable on $(H, \mathcal{N}(0, tQ))$, i.e. it is normally distributed with mean 0 and covariance $|h|^2 \;\; t > 0$. Hence the map: $H \to L^2(H, \mathcal{N}(0, tQ))$, $h \mapsto < (tQ)^{-1/2}\,(\cdot), h >$ is a linear isometry.

We recall the definition of the *heat semigroup* O_t on $\mathcal{C}_b(H)$ (see also (2.1.3)):

$$O_t f(x) = \int_H f(x+y)\, \mathcal{N}(0, tQ) dy, \quad f \in \mathcal{C}_b(H), \; x \in H, \; t \geq 0. \tag{3.2.3}$$

It turns out that O_t is a semigroup of bounded linear operators on $\mathcal{C}_b(H)$. Moreover it is well known, see for instance Gross [41] or Theorem II.6.1 in Kuo [50], that O_t, is a strongly continuous semigroup on $\mathcal{C}_b(H)$. The *(infinitesimal) generator* of O_t will be denoted by $\mathcal{A}$ (as concerns the main concepts on strongly continuous semigroups of bounded linear operators, or briefly $\mathcal{C}_0$-semigroups, we refer to Pazy [61]).

When $H = \mathbb{R}^n$, it is well known that O_t is an analytic semigroup on $\mathcal{C}_b(\mathbb{R}^n)$. We are going to show that the situation is completely different when H is infinite dimensional.

To this purpose let us review that a $\mathcal{C}_0$-semigroup P_t on a Banach space X is *eventually differentiable,* respectively *eventually norm continuous,* if there exists a $t_0 \geq 0$ such that the map $t \mapsto P_t$, is differentiable, respectively continuous, from (t_0, ∞) into $\mathcal{L}(X)$, endowed with the norm operator topology.

In Guiotto [42] it is proved that if H has infinite dimension then O_t is not *eventually differentiable* in $\mathcal{C}_b(H)$. Of course this implies in particular that O_t is not analytic in $\mathcal{C}_b(H)$.

Later in Desch and Rhandi [27] it is shown that if H has infinite dimension then O_t is not eventually norm continuous. Recently this result has been obtained in Van Neerven and Zabczyk [82], with a different approach that also allows to prove discontinuity results for the more general class of Ornstein-Uhlenbeck semigroups.

We present now the proof in Van Neerven and Zabczyk [82] about the norm discontinuity of O_t. This result will be used in the sequel.

Theorem 3.2.1 *Let O_t be the heat semigroup on $\mathcal{C}_b(H)$, where H is an infinite dimensional, real separable Hilbert space. Then it holds:*

$$\|O_{t+h} - O_t\|_{\mathcal{L}(\mathcal{C}_b(H))} = 2, \quad t \geq 0,\, h > 0.$$

Proof First of all we set $\mathcal{N}(0, tQ) = p_t$, $t \geq 0$. Now considering the dual semigroup O'_t of O_t, there results

$$\begin{aligned} < O'_t(p_s), f >=< p_s, O_t f >= \int_H \int_H f(z+y)\, p_t(dy) p_s(dz) \\ = \int_H f(u)\, (p_t * p_s)(du) = \int_H f(u)\, p_{t+s}(du) \\ =< p_{t+s}, f >, \quad f \in \mathcal{C}_b(H),\ t,\, s \geq 0 \end{aligned}$$

where $< \cdot, \cdot >$ denotes the duality pairing between $\mathcal{C}_b(H)$ and its topological dual $\mathcal{C}_b(H)'$. Of course any finite signed Borel measure on H can be considered as an element of $\mathcal{C}_b(H)'$. In the last formula we have proved that $O'_t(p_s) = p_{t+s}$ in $\mathcal{C}_b(H)'$.

Now denote by $\mathrm{var}(p_t - p_s)$, the total variation of the signed Borel measure $p_t - p_s$ (see for instance Ash [4] for details on signed measures). We use the following simple fact: for any finite signed Borel measure μ on H we have that $\mathrm{var}\ (\mu) = \|\mu\|_{\mathcal{C}_b(H)'}$ (for a more general result we refer to the proof of Theorem 6.2.3). There results, for any $t, s \geq 0$ such that $t \neq s$, $h > 0$,

$$\|O_{t+h} - O_t\|_{\mathcal{L}(\mathcal{C}_b(H))} = \|O'_{t+h} - O'_t\|_{\mathcal{L}(\mathcal{C}_b(H)')} \geq \|O'_{t+h}(p_s) - O'_t(p_s)\|_{\mathcal{C}_b(H)'}$$

$$= \|p_{t+s+h} - p_{t+s}\|_{\mathcal{C}_b(H)'} = \mathrm{var}\ (p_{t+s+h} - p_{t+s}) = 2.$$

In the last passage we have used that p_t and p_s are mutually singular since $t \neq s$. The proof is complete. ∎

In the remainder of this section we discuss the regularization effects of the heat semigroup in $\mathcal{C}_b(H)$.

If $H = \mathbb{R}^n$, it is well known that for any $f \in \mathcal{C}_b(\mathbb{R}^n)$ one has $O_t f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, $t > 0$. However when H is infinite dimensional, for any $t > 0$, there exists a map $\hat{f} \in \mathcal{C}_b(H)$ such that $O_t \hat{f}$ is not a Lipschitz continuous map (see Proposition 2.2.5).

By using the Cameron-Martin formula (see (3.2.2)), it is possible to prove that for any $f \in \mathcal{C}_b(H)$, there results that $O_t f$ is differentiable along the reproducing kernel space of $\mathcal{N}(0, Q)$. More precisely if (X, H_0, p_1) is an abstract Wiener space (see §1.1.1), the following statement holds:

$$O_t f \in \mathcal{C}^\infty_{H_0}(X), \quad f \in \mathcal{C}_b(X), \quad t > 0 \tag{3.2.4}$$

and suitable estimates on the H_0-derivatives of $O_t f$ are given (see Proposition 9 in Gross [41] and Elson [32, §3.1]).

If $X = H$ and $H_0 = Q^{1/2}H$, using Proposition 1.3.2, the statement (3.2.4) is equivalent to the following result.

Proposition 3.2.2 *Let $f \in \mathcal{C}_b(H)$, then $O_t f \in \mathcal{C}_Q^\infty(H)$, $t > 0$, with the first and second derivatives given by*

$$< D_Q O_t f(x), v >= \frac{1}{\sqrt{t}} \int_H f(x+y) < (tQ)^{-1/2} y, v > \mathcal{N}(0, tQ) dy,$$

$$\begin{aligned} &< D_Q^2 O_t f(x) u, v >= \frac{1}{t} \int_H f(x+y) < (tQ)^{-1/2} y, u >< (tQ)^{-1/2} y, v > \mathcal{N}(0, tQ) dy \\ &-\frac{1}{t} O_t f(x) < u, v >, \quad u,\ v,\ x \in H,\ t > 0. \end{aligned} \tag{3.2.5}$$

Moreover one has

$$\|D_Q O_t f\|_0 \le \frac{1}{\sqrt{t}} \|f\|_0, \quad \|D_Q^2 O_t f\|_{0, \mathcal{L}(H)} \le \frac{2}{t} \|f\|_0, \ \ f \in \mathcal{C}_b(H). \tag{3.2.6}$$

Proof We fix $t > 0$ and start to verify that $O_t f \in \mathcal{C}_Q^1(H)$. To this purpose we use the Cameron-Martin formula (see (3.2.2)), setting for convenience, for any $u \in H$,

$$W_u(y) \stackrel{\text{def}}{=} \sqrt{t} < (tQ)^{-1/2} y, u >, \quad y \in H, \mathcal{N}(0, tQ) - \text{a.e.}\ .$$

This way one gets, for any u, $x \in H$,

$$\begin{aligned} &\Big| \frac{O_t f(x + s Q^{1/2} u) - O_t f(x)}{s} - \frac{1}{t} \int_H f(x+y) W_u(y) \mathcal{N}(0, tQ) dy \Big| \\ &= \Big| \int_H f(x+y) \Big(\frac{\exp\left[-\frac{1}{2t} s^2 |u|^2 + \frac{1}{t} s W_u(y) \right] - 1}{s} - \frac{W_u(y)}{t} \Big) \mathcal{N}(0, tQ) dy \Big| \\ &\le \|f\|_0 \int_H \Big| \frac{1}{s} \Big(\exp\left[-\frac{1}{2t} s^2 |u|^2 + \frac{1}{t} s W_u(y) \right] - 1 \Big) - \frac{W_u(y)}{t} \Big| \mathcal{N}(0, tQ) dy. \end{aligned} \tag{3.2.7}$$

Now consider the following estimate, for any $v \in H$,

$$\begin{aligned} &\Big| \frac{1}{s} \Big(\exp\Big[-\frac{1}{2t} s^2 |v|^2 + \frac{s}{t} W_v(y) \Big] - 1 \Big) \Big| \\ &\le \exp\Big[\frac{1}{t} |W_v(y)| \Big] \Big(\frac{|v|^2 + |W_v(y)|}{t} \Big), \ \ 0 < s < 1,\ y \in H. \end{aligned}$$

Notice that the map $\exp[\frac{1}{t}|W_v(\cdot)|]$ is $\mathcal{N}(0, tQ)$-integrable, since W_v is a Gaussian random variable on $(H, \mathcal{N}(0, tQ))$ with distribution $\mathcal{N}(0, t|v|^2)$.

Letting $s \to 0^+$ in the last term of (3.2.7), by the Dominated Convergence Theorem, we obtain that there exists the partial derivative of $O_t f$ at $x \in H$, with respect to $Q^{1/2} u$. Moreover, by using the Hölder inequality, we infer, for any x, $u \in H$,

$$\begin{aligned} &\Big(\frac{1}{t} \int_H f(x+y) W_u(y) \mathcal{N}(0, tQ) dy \Big)^2 \le \frac{1}{t^2} \|f\|_0^2 \int_H |W_u(y)|^2 \mathcal{N}(0, tQ) dy \\ &\le \frac{1}{t} \|f\|_0^2 \, |u|^2. \end{aligned} \tag{3.2.8}$$

From this estimate we deduce that there exists $D_Q O_t f(x) \in H$ and is given by (3.2.5). Moreover, arguing as in (3.2.8), one obtains

$$
\begin{aligned}
&| < D_Q O_t f(x) - D_Q O_t f(z), u > |^2 \leq \frac{1}{t^2}\omega_f(|x-z|)^2 \int_H |W_u(y)|^2 \, \mathcal{N}(0,tQ)dy \\
&\leq \frac{1}{t}\,\omega_f(|x-z|)^2\,|u|^2, \quad x,\, z \in H,\ t>0,
\end{aligned}
\tag{3.2.9}
$$

where ω_f is the modulus of continuity of f. By (3.2.8) and (3.2.9) it follows that $D_Q O_t f \in \mathcal{C}_b(H,H)$ and so $O_t f \in \mathcal{C}^1_Q(H)$. By (3.2.8) we also deduce the first estimate in (3.2.6).

Now we prove that there exists $D^2_Q O_t f(x)$, $x \in H$. Appealing again to the Cameron-Martin formula we obtain, for any $s \in (0,1]$, u, v, $x \in H$,

$$
\begin{aligned}
&\frac{< D_Q O_t f(x+sQ^{1/2}v) - D_Q O_t f(x), u >}{s} \\
&= \frac{1}{t}\int_H \frac{f(x+y+sQ^{1/2}v) - f(x+y)}{s}\, W_u(y)\mathcal{N}(0,tQ)dy \\
&= \frac{1}{s}\,\frac{1}{t}\Big[\int_H f(x+y)W_u(y)\,\mathcal{N}(sQ^{1/2}v,tQ)dy \;-\; s < v,u > O_t f(x) \\
&- \int_H f(x+y)\, W_u(y)\mathcal{N}(0,tQ)dy\Big] \\
&= \frac{1}{t}\int_H f(x+y)W_u(y)\Big(\frac{\exp\big[-\frac{1}{2t}s^2|v|^2 + \frac{1}{t}sW_v(y)\big] - 1}{s}\Big)\mathcal{N}(0,tQ)dy \\
&- \frac{< u,v >}{t} O_t f(x) \;\stackrel{\text{def}}{=}\; I_{s,u,t,x}.
\end{aligned}
\tag{3.2.10}
$$

Now we find, applying the Dominated Convergence Theorem,

$$
\begin{aligned}
&\lim_{s\to 0^+}\ \sup_{|u|=1}\ \Big| I_{s,u,t,x} \\
&- \frac{1}{t^2}\int_H f(x+y)\,W_u(y)W_v(y)\,\mathcal{N}(0,tQ)dy - \frac{< u,v >}{t} O_t f(x)\Big| \;=\; 0.
\end{aligned}
\tag{3.2.11}
$$

Moreover, using the Hölder inequality and the fact that W_h, $h \in H$, has Gaussian distribution $\mathcal{N}(0,t|h|^2)$ on $\mathbb{R}$, we have

$$
\begin{aligned}
&\int_H |W_u(y)|^2\,|W_v(y)|^2\,\mathcal{N}(0,tQ)dy \\
&\leq \Big(\int_H |W_u(y)|^4\mathcal{N}(0,tQ)dy\Big)^{1/2}\,\Big(\int_H |W_v(y)|^4\mathcal{N}(0,tQ)dy\Big)^{1/2} \\
&=\; 3t^2\,|u|^2|v|^2, \quad u,v \in H.
\end{aligned}
\tag{3.2.12}
$$

Combining (3.2.11) and (3.2.12) we find readily that there exists $D^2_Q O_t f(x) \in \mathcal{L}(H)$. Then, proceeding as in (3.2.8) and (3.2.9), it follows that $O_t f \in \mathcal{C}^2_Q(H)$ and formula (3.2.5) holds. It remains to check the second estimate in (3.2.6).

To this purpose we set $O_t f = O_{t/2} O_{t/2} f$ so that it holds

$$< D^2_Q O_t f(x)u, v >= \frac{\sqrt{2}}{\sqrt{t}} \int_H < D_Q O_{t/2} f(x+y), v >< (\tfrac{t}{2}Q)^{-1/2} y, u > \mathcal{N}(0, \tfrac{t}{2}Q) dy, \tag{3.2.13}$$

where u, v, $x \in H$. Indeed notice that by (3.2.10) one infers, for any $g \in \mathcal{C}^1_Q(H)$,

$$< D^2_Q O_t g(x)u, v >= \frac{1}{\sqrt{t}} \int_H < D_Q g(x+y), v >< (tQ)^{-1/2} y, u > \mathcal{N}(0, tQ) dy. \tag{3.2.14}$$

Applying the Hölder inequality in (3.2.13) and the first estimate of (3.2.6), there results

$$\begin{aligned}
&| < D^2_Q O_t f(x)u, v > |^2 \le \left| \frac{\sqrt{2}}{\sqrt{t}} \int_H < D_Q O_{t/2} f(x+y), u >< (\tfrac{t}{2}Q)^{-1/2} y, v > \mathcal{N}(0, \tfrac{t}{2}Q) dy \right|^2 \\
&\le \frac{2}{t} \|D_Q O_{t/2} f\|_0^2 \, |u|^2 \int_H \left| < (\tfrac{t}{2}Q)^{-1/2} y, v > \right|^2 \mathcal{N}(0, \tfrac{t}{2}Q) dy \\
&\le \ \frac{4}{t^2} \, |v|^2 \, |u|^2 \, \|f\|_0^2.
\end{aligned}$$

Taking the supremum, in the last formula, over all u, $v \in H$, $|u| = 1$, $|v| = 1$, we find the second estimate of (3.2.6).

In order to prove that $O_t f \in \mathcal{C}^n_Q(H)$, $n \ge 3$, we can perform the same technique, using the integrability properties of the Gaussian random variable $< (tQ)^{-1/2} \cdot, u >$, $u \in H$. However the computation is more involved. ■

The next result deals with the problem of the summability for the linear bounded operator $D^2_Q O_t f(x)$, $x \in H$. It can be proved following the proof of Proposition 9 in Gross [41] (see also Proposition 8 in Piech [64]). However we present here a direct and simpler proof.

Proposition 3.2.3 *For any $f \in \mathcal{C}_b(H)$, we have that $D^2_Q O_t f(x)$ is of Hilbert-Schmidt type for any $x \in H$, $t > 0$. Moreover $D^2_Q O_t f \in \mathcal{C}_b(H, \mathcal{L}_2(H))$, $t > 0$ and it holds:*

$$\begin{aligned}
&(i) \quad \sup_{x \in H} \|D^2_Q O_t f(x)\|_{\mathcal{L}_2(H)} = \|D^2_Q O_t f\|_{0,\mathcal{L}_2(H)} \ \le \ \frac{2}{t} \|f\|_0, \ f \in \mathcal{C}_b(H), \\
&(ii) \ \ \|D^2_Q O_t g\|_{0,\mathcal{L}_2(H)} \ \le \ \frac{1}{\sqrt{t}} \|g\|_{1,Q}, \ \ g \in \mathcal{C}^1_Q(H), \quad t > 0.
\end{aligned}$$

Proof Let $f \in \mathcal{C}_b(H)$ and fix $t > 0$. We set $O_t f = O_{t/2} O_{t/2} f$. By formula (3.2.13), we know that

$$< D_Q^2 O_t f(x)u, v >= \frac{\sqrt{2}}{\sqrt{t}} \int_H < D_Q O_{t/2} f(x+y), v >< (\tfrac{t}{2}Q)^{-1/2} y, u > \mathcal{N}(0, \tfrac{t}{2}Q)dy, \tag{3.2.15}$$

where $u,\ v,\ x \in H$. We want to apply Lemma 1.1.3 in order to obtain that $D_Q^2 O_t f(x) \in \mathcal{L}_2(H)$, $x \in H$. To this end denote by $\mathcal{F}_1$, the set of all finite rank operators in $\mathcal{L}(H)$ such that $\|N\|_{\mathcal{L}_2(H)} \le 1$. We fix any $N \in \mathcal{F}_1$.

In $N(H)$ we choose an othonormal basis $(l_k), k = 1, ..n$. Then we set, for convenience,

$$< (\tfrac{t}{2}Q)^{-1/2} y, u >= J_u(y),\ u \in H,\ y \in H,\ \mathcal{N}(0, tQ) - a.e..$$

Moreover N^* stands for the adjoint of N. Applying first the Hölder inequality and then the Schwarz inequality we can deduce from (3.2.15):

$$\begin{aligned}
&\Big|\mathrm{Tr}\ (ND_Q^2 O_t f(x))\Big|^2 = \Big|\sum_{k=1}^n < D_Q^2 O_t f(x)(l_k), N^* l_k >\Big|^2 \\
&= \Big|\sum_{k=1}^n \frac{\sqrt{2}}{\sqrt{t}} \int_H < D_Q O_{t/2} f(x+y), l_k >< (\tfrac{t}{2}Q)^{-1/2} y, N^* l_k > \mathcal{N}(0, \tfrac{t}{2}Q)dy\Big|^2 \\
&\le \frac{2}{t} \int_H \Big|\sum_{k=1}^n < D_Q O_{t/2} f(x+y), l_k >\ J_{N^* l_k}(y)\Big|^2\ \mathcal{N}(0, \tfrac{t}{2}Q)dy \\
&\le \frac{2}{t} \int_H \left(\sum_{k=1}^n | < D_Q O_{t/2} f(x+y), l_k > |^2\right) \left(\sum_{k=1}^n |J_{N^* l_k}(y)|^2\right) \mathcal{N}(0, \tfrac{t}{2}Q)dy \\
&\le \frac{2}{t}\|D_Q O_{t/2} f\|_0^2 \sum_{k=1}^n \int_H |J_{N^* l_k}(y)|^2 \mathcal{N}(0, \tfrac{t}{2}Q)dy\ =\ \frac{2}{t}\|D_Q O_{t/2} f\|_0^2 \sum_{k=1}^n |N^* l_k|^2 \\
&= \frac{2}{t}\ \|D_Q O_{t/2} f\|_0^2\ \|N^*\|_2^2\ =\ \frac{2}{t}\ \|D_Q O_{t/2} f\|_0^2, \quad x \in H,\ t > 0.
\end{aligned} \tag{3.2.16}$$

Now using formula (3.2.6) it follows

$$\Big|\mathrm{Tr}\ (ND_Q^2 O_t f(x))\Big| \le \frac{\sqrt{2}}{\sqrt{t}}\frac{\sqrt{2}}{\sqrt{t}}\ \|f\|_0 = \frac{2}{t}\|f\|_0.$$

Taking the supremum over all $N \in \mathcal{F}_1$ and invoking Lemma 1.1.3, we have $D_Q^2 O_t f(x) \in \mathcal{L}_2(H)$ and $\|D_Q^2 O_t f\|_{0,\mathcal{L}_2(H)} \le \frac{2}{t}\|f\|_0$.

To verify the uniform continuity of $D_Q^2 O_t f$, we proceed as in (3.2.16) in order to obtain, for any $x,\ z \in H,\ N \in \mathcal{F}_1$,

$$\Big|\mathrm{Tr}\ (N[D_Q^2 O_t f(x) - D_Q^2 O_t f(z)])\Big| \le \frac{2}{t}\omega_f(|x-z|),\ \ x,\ z \in H.$$

Taking the supremum over all $N \in \mathcal{F}_1$ and invoking Lemma 1.1.3, we find

$$\|D_Q^2 O_t f(x) - D_Q^2 O_t f(z)\|_{\mathcal{L}_2(H)} \leq \frac{2}{t}\omega_f(|x-z|)$$

and the uniform continuity follows. To deduce (ii), we start from

$$< D_Q^2 O_t g(x)u, v >= \frac{1}{\sqrt{t}} \int_H < D_Q g(x+y), v >< (tQ)^{-1/2}y, u > \mathcal{N}(0, tQ)dy,$$

where $g \in \mathcal{C}_Q^1(H)$ and then we proceed as in (3.2.16). The proof is complete. ■

Remark 3.2.4 We point out that Proposition 3.2.3 can be generalized by introducing the space $\mathcal{L}_2^n(H)$ of all symmetric n-linear Hilbert-Schmidt functionals on $H \times ... \times H$ (n-times), $n \geq 2$. Let $T \in \mathcal{L}^n(H)$, one has:

$$T \in \mathcal{L}_2^n(H) \iff \sum_{i_1,\dots,i_n=1}^{\infty} T(e_{i_1}, \dots, e_{i_n})^2 < \infty, \tag{3.2.17}$$

where (e_k) is an orthonormal basis of H. The sum (3.2.17) is independent of (e_k). For any $n \geq 2$ it turns out that $\mathcal{L}_2^n(H)$ is a Hilbert space endowed with the inner product

$$< S, T >_{\mathcal{L}_2^n(H)} \stackrel{\text{def}}{=} \sum_{i_1,\dots,i_n=1}^{\infty} T(e_{i_1}, \dots, e_{i_n})\, S(e_{i_1}, \dots, e_{i_n}), \;\; T,\, S \in \mathcal{L}_2^n(H).$$

Now we are ready to state the following result (see Proposition 3.2 in Lee [53] and also Piech [64] and Kuo [51]):

for any $f \in \mathcal{C}_b(H)$ we have that $D_Q^n O_t f(x)$ belongs to $\mathcal{L}_2^n(H)$, $x \in H$, $t > 0$. Moreover $D_Q^n O_t f \in \mathcal{C}_b(H, \mathcal{L}_2^n(H))$, $n \geq 2$, $t > 0$. We shall not use such a difficult result, that requires involved computations. Thus we omit further details. ▪

3.3 A new characterization for the generator of O_t

This section is mainly concerned with the connections between the heat semigroup O_t in $\mathcal{C}_b(H)$, with infinitesimal generator denoted by $\mathcal{A}$, and the linear operator $\mathcal{A}_1$, defined as follows

$$\begin{cases} D(\mathcal{A}_1) = \; \{f \in \mathcal{C}_Q^2(H) \text{ such that } D_Q^2 f(x) \in \mathcal{L}_1(H), \;\; x \in H \text{ and} \\ D_Q^2 f \in \mathcal{C}_b(H, \mathcal{L}_1(H))\}; \\ \\ \mathcal{A}_1 : D(\mathcal{A}_1) \;\to\; \mathcal{C}_b(H), \;\; \mathcal{A}_1 f(x) \stackrel{\text{def}}{=} \frac{1}{2}\,\text{Tr}\,[D_Q^2 f(x)], \; f \in D(\mathcal{A}_1), \; x \in H. \end{cases} \tag{3.3.1}$$

The operator $\mathcal{A}_1$ was introduced in Gross [41]. In terms of the orthonormal basis $\{e_k\}_{k\geq 1}$, that diagonalizes Q, we can write for any $f \in D(\mathcal{A}_1)$,

$$\mathcal{A}_1 f(x) \;=\; \frac{1}{2}\mathrm{Tr}\,[D_Q^2 f(x)] \;=\; \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k D_{kk} f(x), \quad x \in H.$$

The next simple proposition shows that $D(\mathcal{A}_1)$ is invariant for O_t.

Proposition 3.3.1 *Let $f \in D(\mathcal{A}_1)$, then for any $t > 0$ one has $O_t f \in D(\mathcal{A}_1)$.*

Proof Let $f \in D(\mathcal{A}_1)$, we fix $t > 0$ and write

$$O_t f(x) = \int_H f(x+y)\,\mathcal{N}(0,tQ)dy, \;\; x \in H.$$

Since $f \in \mathcal{C}_Q^2(H)$, applying the Dominated Convergence Theorem, we can differentiate under the integral sign and obtain

$$< D_Q^2 O_t f(x)u, v >= \int_H < D_Q^2 f(x+y)u, v > \mathcal{N}(0,tQ)dy,$$

where x, u, v, $\in H$. Moreover since $D_Q^2 f \in \mathcal{C}_b(H, \mathcal{L}_1(H))$, we can write

$$D_Q^2 O_t f(x) \;=\; \int_H D_Q^2 f(x+y)\,\mathcal{N}(0,tQ)dy, \;\; x \in H, \tag{3.3.2}$$

where the integral has to be understood in the Bochner sense. This way it is clear that $D_Q^2 O_t f \in \mathcal{C}_b(H, \mathcal{L}_1(H))$. ■

Now we present the main result of the chapter: it asserts that $D(\mathcal{A}_1)$ is a core for the generator $\mathcal{A}$ of O_t.

This result was announced in Cannarsa and Da Prato [13] (see Proposition 4.1) but there the proof was not complete. It was proved in Priola [68]. It extends a classical theorem due to Gross (see Theorem 3 and Corollary 3.2 of Gross [41]) and gives a new contribution to the long standing problem of characterizing $\mathcal{A}$.

Theorem 3.3.2 *Let O_t be the heat semigroup on $\mathcal{C}_b(H)$ defined by (3.2.3) with generator $\mathcal{A}$. Then*

(i) $\mathcal{A}$ is an extension of $\mathcal{A}_1$;

(ii) $D(\mathcal{A}_1)$ is dense in $D(\mathcal{A})$ with respect to the graph norm.

Proof Statement (ii) follows by (i) and by Corollary 3.2 in Gross [41] which asserts that $D(\mathcal{A}_1) \cap D(\mathcal{A})$ is dense in $D(\mathcal{A})$ with respect to the graph norm (for a different and simpler proof of (ii) we refer to Theorem 3.3.5). Let us prove the first assertion.

(i) Fix $\hat{f} \in D(\mathcal{A}_1)$. We have to prove that $\hat{f} \in D(\mathcal{A})$ and that $\mathcal{A}\hat{f}(x) = \mathcal{A}_1 \hat{f}(x), \quad x \in H.$
We split up the proof into several steps.

Step 1. Denote by $P^n : H \to \mathbb{R}^n$, $n \geq 1$, the finite dimensional projections with respect to $\{e_k\}_{k\geq 1}$ (see (3.2.1)):

$$P^n x \ = \ \sum_{k=1}^{n} x_k e_k, \quad x \in H.$$

Let us introduce for any $n \geq 1$, $t > 0$, the following approximating operators O_t^n : $\mathcal{C}_b(H) \to \mathcal{C}_b(H)$, defined as follows

$$O_t^n f(x) \ = \ \int_H f(x + P^n y)\, \mathcal{N}(0, tQ) dy, \quad f \in \mathcal{C}_b(H), \ x \in H. \tag{3.3.3}$$

It is easy to check, using standard properties of Gaussian measures, that for any $n \geq 1$, O_t^n is a strongly continuous semigroup of bounded linear operators on $\mathcal{C}_b(H)$. We give now a proof of the fact that $O_t^n f \to O_t f$ in $\mathcal{C}_b(H)$ as $n \to \infty$, uniformly in t on bounded sets of $[0, \infty[$ (a similar statement was proved in Theorem 3.1 in Cannarsa and Da Prato [13]).

Let $f \in \mathcal{C}_b(H)$ and denote by ω_f the modulus of continuity of f. We have for any $n \geq 1$, $T > 0$, $f \in \mathcal{C}_b(H)$, $t \in [0, T]$,

$$\begin{aligned} &\sup_{x\in H} |O_t^n f(x) \ - \ O_t f(x)| \\ &\leq \sup_{x\in H} \int_H |f(x + \sqrt{t} P^n y) - f(x + \sqrt{t} y)|\, \mathcal{N}(0, Q) dy \\ &\leq \ \int_H \omega_f(\sqrt{t} |P^n y - y|)\, \mathcal{N}(0, Q) dy \ \leq \ \int_H \omega_f(\sqrt{T} |P^n y - y|)\, \mathcal{N}(0, Q) dy. \end{aligned} \tag{3.3.4}$$

Let us notice that the map: $H \to \mathbb{R}$, $y \mapsto \omega_f(|y|)$ is continuous and bounded. Now letting $n \to \infty$ in the last term of (3.3.4), we obtain the assertion by the Dominated Convergence Theorem.

Step 2. We verify that

$$\frac{d}{dt}\, O_t^n \hat{f}(x) \ = \ \tfrac{1}{2} O_t^n \Big(\sum_{k=1}^{n} \lambda_k D_{kk} \hat{f} \Big)(x), \quad x \in H, \ t \geq 0.$$

First note that it holds for any $x \in H$, $t > 0$, $n \geq 1$,

$$\begin{aligned} O_t^n \hat{f}(x) \ &= \ \int_H \hat{f}(x + P^n y)\, \mathcal{N}(0, t\, P^n Q) dy \\ &= \ \frac{1}{\sqrt{(2\pi)^n\, \lambda_1 \ldots \lambda_n\, t^n}} \int_{\mathbb{R}^n} \hat{f}(x + \sum_{i=1}^{n} y_i e_i)\ e^{-\frac{1}{2t} \sum_{i=1}^{n} \frac{y_i^2}{\lambda_i}}\ dy_1 \ldots dy_n. \end{aligned} \tag{3.3.5}$$

Fix $x \in H$, differentiating with respect to t in (3.3.5), using standard properties of Gaussian measures, we obtain

$$
\begin{aligned}
\frac{d}{dt}\,O_t^n\hat{f}(x) &= \sum_{k=1}^{n}\frac{1}{2\sqrt{t}}\int_H D_k\hat{f}(x+\sqrt{t}P^n y)y_k\,\mathcal{N}(0,Q)dy\\
&= \frac{1}{2}\sum_{k=1}^{n}\lambda_k D_k\Big(\int_H D_k\hat{f}(x+P^n y)\,\mathcal{N}(0,tQ)dy\Big)\\
&= \frac{1}{2}\sum_{k=1}^{n}\lambda_k\int_H D_{kk}\hat{f}(x+P^n y)y_k\,\mathcal{N}(0,tQ)dy\\
&= O_t^n\Big(\frac{1}{2}\sum_{k=1}^{n}\lambda_k D_{kk}\hat{f}\Big)(x),\quad t\geq 0,\ n\geq 1.
\end{aligned}
\tag{3.3.6}
$$

Step 3. We set $F_n(x) = \sum_{k=1}^{n}\lambda_k D_{kk}\hat{f}(x)$, $n\geq 1$, $x\in H$ and we prove that (F_n) is a sequence of uniformly bounded and equi-uniformly continuous functions on $\mathcal{C}_b(H)$.

We use that for any $T\in\mathcal{L}_1(H)$, $A\in\mathcal{L}(H)$, it holds $TA\in\mathcal{L}_1(H)$ and moreover

$$
|\mathrm{Tr}\,(TA)| \leq \|TA\|_1 \leq \|A\|_{\mathcal{L}(H)}\,\|T\|_1. \tag{3.3.7}
$$

By hypothesis $D_Q^2\hat{f}\in\mathcal{C}_b(H,\mathcal{L}_1(H))$ and we denote by $\omega_{D_Q^2\hat{f}}$ the modulus of continuity of $D_Q^2\hat{f}$. Moreover we set $\sup_{x\in H}\|D_Q^2\hat{f}(x)\|_1 = \|D_Q^2\hat{f}\|_0$. Using (3.3.7) we find

$$
\begin{aligned}
|F_n(x)| &= |\sum_{k=1}^{n}< P^n D_Q^2\hat{f}(x)e_k, e_k >| = |\mathrm{Tr}\,(P^n D_Q^2\hat{f}(x))|\\
&\leq \|P^n\|_{\mathcal{L}(H)}\,\|D_Q^2\hat{f}(x)\|_1 \leq \|D_Q^2\hat{f}\|_0,\quad x\in H.
\end{aligned}
\tag{3.3.8}
$$

Therefore (F_n) is uniformly bounded. The equicontinuity follows from the inequalities

$$
\begin{aligned}
&|F_n(x)-F_n(z)| = |\mathrm{Tr}\,(P^n\,[D_Q^2\hat{f}(x)-D_Q^2\hat{f}(z)])|\\
&\leq \|P^n\|_{\mathcal{L}(H)}\|D_Q^2\hat{f}(x)-D_Q^2\hat{f}(z)\|_1 \leq \omega_{D_Q^2\hat{f}}(|x-z|),\quad x,\,z\in H,\ n\geq 1.
\end{aligned}
\tag{3.3.9}
$$

Step 4. We verify that

$$
\frac{d}{dt}\,O_t\hat{f}(x) = O_t\mathcal{A}_1\hat{f}(x)\quad t\geq 0,\ x\in H. \tag{3.3.10}
$$

Fix $x\in H$ and consider $O_t^n\hat{f}(x)$ as a function of t, $n\geq 1$. By the first step we know in particular that

$$
\lim_{n\to\infty} O_t^n\hat{f}(x) = O_t\hat{f}(x)\quad t\geq 0. \tag{3.3.11}
$$

Moreover by the second and third step, we have that for any $n\geq 1$,

$$|\frac{d}{dt}\, O_t^n \hat{f}(x)| \;=\; \frac{1}{2}|O_t^n(F_n)(x)| \;\le\; \frac{1}{2}\|D_Q^2 \hat{f}\|_0, \;\; t \ge 0. \tag{3.3.12}$$

Note that once we have proved that

$$\lim_{n\to\infty}\, \frac{d}{dt}\, O_t^n \hat{f}(x) \;=\; O_t(\mathcal{A}_1 f)(x), \quad t \ge 0, \tag{3.3.13}$$

since $O_{(\cdot)}^n\, \hat{f}(x) \in \mathcal{C}^1([0,\infty[)$ for any $n \ge 1$, $O_{(\cdot)}\mathcal{A}_1 \hat{f}(x) \in \mathcal{C}([0,\infty[)$ and the estimate (3.3.12) holds, applying a well known lemma of Real Analysis, we will be able to conclude that $O_{(\cdot)}\, \hat{f}(x) \in \mathcal{C}^1([0,\infty[)$ and that (3.3.10) holds.

Let us check assertion (3.3.13). For any $t \ge 0$, $n \ge 1$, we infer

$$\begin{aligned}
&\Big|\frac{d}{dt}\, O_t^n \hat{f}(x) \;-\; O_t\mathcal{A}_1 \hat{f}(x)\Big| \;=\; \Big|\frac{1}{2} O_t^n F_n(x) - O_t(\mathcal{A}_1 \hat{f})(x)\Big| \\
&\le \frac{1}{2}\Big|O_t^n F_n(x) - O_t(F_n)(x)\Big| \;+\; \Big|O_t\,[\frac{1}{2}F_n(x) - \mathcal{A}_1 \hat{f}](x)\Big| \\
&\le \frac{1}{2}\int_H \Big|F_n(x+P^n y) \;-\; F_n(x+y)\Big|\, \mathcal{N}(0,tQ)dy \\
&+ \frac{1}{2}\int_H \Big|\sum_{k=n+1}^{\infty} \lambda_k D_{kk}\hat{f}(x+y)\Big|\, \mathcal{N}(0,tQ)dy \\
&\le \frac{1}{2}\int_H \Big[\omega_{D_Q^2 \hat{f}}\,(|P^n y - y|) \;+\; \sum_{k=n+1}^{\infty} \lambda_k D_{kk}\hat{f}(x+y)\Big]\, \mathcal{N}(0,tQ)dy.
\end{aligned} \tag{3.3.14}$$

Now, similarly to (3.3.8), we have for any $z \in H$, $n \ge 1$,

$$\Big|\sum_{k=n+1}^{\infty} \lambda_k D_{kk}\hat{f}(z)\Big| \;=\; |\mathrm{Tr}\;([I - P_n]\, D_Q^2 \hat{f}(z))| \le\; \|D_Q^2 \hat{f}\|_0$$

Letting $n \to \infty$ in the last term of (3.3.14), by the Dominated Convergence Theorem, we deduce (3.3.13). The proof of step 4 is complete.

Step 5. We show that $\hat{f} \in D(\mathcal{A})$ and $\mathcal{A}\hat{f} = \mathcal{A}_1 \hat{f}$.

By step 4 we know that for any $t \ge 0$,

$$O_t \hat{f}(x) - \;\hat{f}(x) \;=\; \int_0^t O_s \mathcal{A}_1 \hat{f}(x) ds, \;\; x \in H.$$

Hence we can write for any $t > 0$, $x \in H$,

$$\Big|\frac{O_t \hat{f}(x) - \hat{f}(x)}{t} - \mathcal{A}_1 \hat{f}(x)\Big| = \;\frac{1}{t}\Big|\int_0^t (O_s \mathcal{A}_1 \hat{f}(x) - \mathcal{A}_1 \hat{f}(x)) ds\Big|.$$

Making the supremum over $x \in H$, we readily find

$$\Big\|\frac{O_t \hat{f} - \hat{f}}{t} \;- \mathcal{A}_1 \hat{f}\,\Big\|_0 \le\; \frac{1}{t}\,\int_0^t \|O_s \mathcal{A}_1 \hat{f} \;- \mathcal{A}_1 \hat{f}\|_0\, ds, \;\; t \ge 0.$$

Letting $t \to 0^+$ in the right hand side and using the strong continuity of O_t yields

$$\lim_{t\to 0^+} \left\| \frac{O_t \hat{f} - \hat{f}}{t} - \mathcal{A}_1 \hat{f} \right\|_0 = 0$$

and so the assertion is verified. The proof is complete. ∎

Let $\mathcal{B}$ be any closed operator on a Banach space X, for any $\lambda \in \rho(\mathcal{B})$, the resolvent set of $\mathcal{B}$, we introduce the *resolvent operator*

$$R(\lambda, \mathcal{B}) = (\lambda I - \mathcal{B})^{-1}. \tag{3.3.15}$$

Now we prove some regularity results for O_t involving the subspace $D(\mathcal{A}_1)$. The first one follows by Theorem 3 in Gross [41]. Here we present a simpler and self-contained proof.

Proposition 3.3.3 *For any $f \in \mathcal{C}_b^1(H)$ the following statements hold:*

(a) $O_t f \in D(\mathcal{A}_1) \cap \mathcal{C}_b^1(H)$ *and*

$$\sup_{x\in H} \|D_Q^2 O_t f(x)\|_{\mathcal{L}_1(H)} \le \frac{\sqrt{\mathrm{Tr}\ (Q)}}{\sqrt{t}} \|Df\|_0, \quad t > 0.$$

(b) $R(\lambda, \mathcal{A}) f \in D(\mathcal{A}_1) \cap \mathcal{C}_b^1(H), \qquad \lambda > 0.$

Proof (a) Fix $f \in \mathcal{C}_b^1(H)$ and $t > 0$. The proof is carried out into two parts.
Step 1. We can verify that $O_t f \in \mathcal{C}_b^1(H)$ and its Fréchet derivative $DO_t f$ is given by

$$DO_t f(x) = \int_H Df(x+y)\, \mathcal{N}(0, tQ) dy, \quad x \in H, \tag{3.3.16}$$

where the integral is intended in the Bochner sense (indeed the Gateâux differentiability of $O_t f$ on H follows by the Dominated Convergence Theorem and, in order to obtain the Fréchet differentiability, one remarks that $DO_t f$ is uniformly continuous from H into H).
Step 2. We already know that $O_t f \in \mathcal{C}_Q^\infty(H)$ (see Proposition 3.2.2). Now we point out that for any $x \in H$, there exists a linear operator $TO_t f(x)$ on H, such that

$$\begin{aligned} &D^2 O_t f(x)(v) = Q^{1/2} TO_t f(x)(v), \quad v \in H \\ &\text{where } \ TO_t f(x)(v) \overset{\text{def}}{=} \frac{1}{\sqrt{t}} \int_H Df(x+y) < (tQ)^{-1/2} y, v > \mathcal{N}(0, tQ) dy. \end{aligned} \tag{3.3.17}$$

This follows by (3.2.14), since $D_Q f(x) = Q^{1/2} Df(x), \quad x \in H$. We verify that $TO_t f(x) \in \mathcal{L}(H)$, $x \in H$. To this purpose we use the following estimate, setting, for any $u \in H$, $< (tQ)^{-1/2} y, u > = J_u(y), \quad y \in H, \mathcal{N}(0, tQ)$-a.e.,

$$\begin{aligned} \|TO_t f(x)(v)\|_H^2 &\le \frac{1}{t} \|Df\|_0^2 \Big(\int_H |J_v(y)|\, \mathcal{N}(0, tQ) dy \Big)^2 \\ &\le \frac{\|Df\|_0^2}{t} \int_H |J_v(y)|^2\, \mathcal{N}(0, tQ) dy \le \frac{\|Df\|_0^2}{t} \|v\|_H^2, \quad x,\ v \in H. \end{aligned}$$

From this estimate we deduce that $TO_tf(x) \in \mathcal{L}(H)$ and further

$$\sup_{x\in H} \|TO_tf(x)\|_{\mathcal{L}(H)} \le \frac{1}{\sqrt{t}}\|Df\|_0. \tag{3.3.18}$$

Step 3. We check that $TO_tf(x) \in \mathcal{L}_2(H)$ for any $x \in H$ and further that

$$TO_tf \in \mathcal{C}_b(H, \mathcal{L}_2(H)). \tag{3.3.19}$$

From this fact it will follow in particular that $D_Q^2O_tf \in \mathcal{C}_b(H, \mathcal{L}_1(H))$.

We proceed similarly to the proof of Proposition 3.2.3. Denote by $\mathcal{F}_1$ the set of all finite rank operators N in $\mathcal{L}(H)$ such that $\|N\|_2 \le 1$. We will use Lemma 1.1.3. To this end we fix $N \in \mathcal{F}_1$. In $N(H)$ we fix an othonormal basis $(l_k), k = 1, ..n$. We can complete (l_k) in order to obtain an orthonormal basis on H. Now applying first the Hölder and then the Schwarz inequality we obtain

$$\begin{aligned}
&|\mathrm{Tr}\ (NTO_tf(x))|^2 = \left|\sum_{k=1}^{n} < TO_tf(x)(l_k), N^*l_k >\right|^2 \\
&\le \frac{1}{t}\int_H \left|\sum_{k=1}^{n} < Df(x+y), l_k > \ J_{N^*l_k}(y)\right|^2 \mathcal{N}(0,tQ)dy \\
&\le \frac{1}{t}\int_H \left(\sum_{k=1}^{n} |< Df(x+y), l_k >|^2\right)\left(\sum_{k=1}^{n} |J_{N^*l_k}(y)|^2\right)\mathcal{N}(0,tQ)dy \qquad (3.3.20) \\
&\le \frac{1}{t}\|Df\|_0^2 \sum_{k=1}^{n} \int_H |J_{N^*l_k}(y)|^2\mathcal{N}(0,tQ)dy = \frac{1}{t}\|Df\|_0^2 \sum_{k=1}^{n} |N^*l_k|^2. \\
&= \frac{1}{t}\|Df\|_0^2.
\end{aligned}$$

Taking the supremum over all $N \in \mathcal{F}_1$, we obtain that $TO_tf(x)$ is of Hilbert-Schmidt type and moreover

$$\sup_{x\in H} \|TO_tf(x)\|_2 \le \frac{1}{\sqrt{t}}\|Df\|_0 \tag{3.3.21}$$

We can repeat the previous computations in order to obtain for any $x,\ z \in H, N \in \mathcal{F}_1$,

$$\begin{aligned}
&|\mathrm{Tr}\ (N[TO_tf(x) - TO_tf(z)])|^2 \le \frac{1}{t}\|Df(x) - Df(z)\|_H^2\,\|N^*\|_2^2. \\
&\text{so that}\quad \|TO_tf(x) - TO_tf(z)\|_2 \le \frac{1}{\sqrt{t}}\|Df(x) - Df(z)\|_H.
\end{aligned} \tag{3.3.22}$$

From these estimates, we find that $TO_tf \in \mathcal{C}_b(H, \mathcal{L}_2(H))$.

Concerning $D_Q^2O_tf = Q^{1/2}TO_tf$ one derives (taking into account that for any $A,\ B \in \mathcal{L}_2(H)$, one has $AB \in \mathcal{L}_1(H)$ and $\|AB\|_1 \le \|A\|_2\|B\|_2$)

$$
\begin{aligned}
&\sup_{x\in H}\|D_Q^2O_tf(x)\|_1 \;\le \frac{1}{\sqrt{t}}\,\|Q^{1/2}\|_2\,\|Df\|_0,\\
&\|D_Q^2O_tf(x)-D_Q^2O_tf(z)\|_1 \le \frac{1}{\sqrt{t}}\|Q^{1/2}\|_2\,\|Df(x)-Df(z)\|_H,\;\; x,\,z\in H.
\end{aligned}
\tag{3.3.23}
$$

Thus we have proved that $D_Q^2O_tf \in \mathcal{C}_b(H,\mathcal{L}_1(H))$. The proof of assertion (a) is complete.

(b) By the Hille-Yosida Theorem we have for any $f\in\mathcal{C}_b(H),\;\lambda>0,\;x\in H$,

$$
R(\lambda,\mathcal{A})f(x) \;=\; \int_0^\infty e^{-\lambda t}O_tf(x)\,dt.
$$

Suppose now that $f\in\mathcal{C}_b^1(H)$. We fix $\lambda>0$ and set $u=R(\lambda,\mathcal{A})f$. Differentiating under the integral sign, using estimates (3.3.23), we readily find that $u\in\mathcal{C}_b^1(H)$ $\cap\,\mathcal{C}_Q^2(H)$ and

$$
< D_Q^2u(x)(u),v >= \int_0^\infty e^{-\lambda t}\; < D_Q^2O_tf(x)(u),v > \;dt,\;\; x,\,u,\,v\in H.
$$

To get that $D_Q^2u(x)\in\mathcal{L}_1(H)$, we use Lemma 1.1.3. Let $N\in\mathcal{G}_1$, the set of all finite rank operators $S\in\mathcal{L}(H)$, such that $\|S\|_{\mathcal{L}(H)}\le 1$. In $N(H)$ we choose an orthonormal basis $(l_k), k=1,..n$. Then there results, by (3.3.23),

$$
\begin{aligned}
|\mathrm{Tr}\;(ND_Q^2u(x))| \;&= \Big|\sum_{k=1}^n < D_Q^2u(x)(l_k),N^*l_k > \Big|\\
&\le \int_0^\infty e^{-\lambda t}\,|\mathrm{Tr}\;(ND_Q^2O_tf(x))|\,dt\\
&\le \sqrt{\mathrm{Tr}\;(Q)}\int_0^\infty e^{-\lambda t}\,t^{-1/2}\,dt = \sqrt{\mathrm{Tr}\;(Q)}\sqrt{\frac{\pi}{\lambda}}.
\end{aligned}
\tag{3.3.24}
$$

Taking the supremum over all $N\in\mathcal{G}_1$, we find that $D_Q^2u(x)$ is of trace class, $x\in H$. In order to obtain that $D_Q^2u\in\mathcal{C}_b(H,\mathcal{L}_1(H))$, we argue as in (3.3.22) with $\mathcal{F}_1$, replaced by $\mathcal{G}_1$. It follows that $R(\lambda,\mathcal{A})f\in\mathcal{C}_b^1(H)\cap D(\mathcal{A}_1),\;\;\lambda>0$. The proof is complete.

■

Now we deal with the space $\mathcal{C}_s^2(H)$ and consider the subspace $Y=D(\mathcal{A}_1)\cap\mathcal{C}_s^2(H)$. By using Theorem 2.2.7, proved in Priola [65], we are going to show that Y is a core for $\mathcal{A}$. We recall that $\mathcal{C}_s^2(H)$ is dense in $\mathcal{C}_b(H)$ and on the contrary $\mathcal{C}_b^2(H)$ is not dense (see Remark 2.2.11). This is why Y will be the least core for $\mathcal{A}$ that we will consider. For any $f\in D(\mathcal{A}_1)\cap\mathcal{C}_s^2(H)$, denoting by $\hat{D}^2f$ the second Hadamard derivative of f, we have, for any $x\in H$,

$$
\begin{aligned}
&D_Q^2f(x)=Q^{1/2}\hat{D}^2f(x)Q^{1/2},\\
&\mathcal{A}_1f(x)=\;\tfrac12\mathrm{Tr}\;(Q^{1/2}\hat{D}^2f(x)Q^{1/2}) =\;\tfrac12\mathrm{Tr}\;(Q\hat{D}^2f(x)).
\end{aligned}
\tag{3.3.25}
$$

We need the following preliminary result.

Proposition 3.3.4 *For any $f \in \mathcal{C}_s^2(H)$, denoting by $\hat{D}^2 f$ the second Hadamard derivative of f, the following statements hold:*
(a) $O_t f \in D(\mathcal{A}_1) \cap \mathcal{C}_s^2(H)$ *and*

$$\sup_{x \in H} \|D_Q^2 O_t f(x)\|_{\mathcal{L}_1(H)} \leq \operatorname{Tr}(Q) \|\hat{D}^2 f\|_{0,\mathcal{L}(H)}, \quad t > 0;$$

(b) $D(\mathcal{A}_1) \cap \mathcal{C}_s^2 H$ *is invariant for* O_t*;*
(c) $R(\lambda, \mathcal{A}) f \in D(\mathcal{A}_1) \cap \mathcal{C}_s^2(H), \quad \lambda > 0.$

Proof **(a)** Fix $t > 0$. We already know by Proposition 3.3.3 that $O_t f \in D(\mathcal{A}_1) \cap \mathcal{C}_b^1(H)$. Let us verify that $O_t f \in \mathcal{C}_s^2(H)$. By (3.3.16), we know that

$$DO_t f(x) = \int_H Df(x+y)\, \mathcal{N}(0, tQ) dy, \quad x \in H. \tag{3.3.26}$$

We prove that $O_t f$ has the second Hadamard derivative on H and that this is given by:

$$\hat{D}^2 O_t f(x)(v) = \int_H \hat{D}^2 f(x+y)(v)\, \mathcal{N}(0, tQ) dy, \quad x,\, v \in H, \tag{3.3.27}$$

where the integral is undertstood in the Bochner sense. Note that the second Hadamard derivative $\hat{D}^2 f$ is not assumed to be continuous with values in $\mathcal{L}(H)$.

Fix $x \in H$ and a compact set K in H and consider the mapping $\Lambda : H \times H \times (0,1] \to H$,

$$\Lambda(y, v, s) \stackrel{\text{def}}{=} \frac{Df(x+y+sv) - Df(x+y)}{s} - \hat{D}^2 f(x+y)(v), \quad y,\, v \in H, \; s \in (0,1].$$

Now (3.3.27) follows by showing that

$$\lim_{s \to 0^+} \sup_{v \in K} \int_H |\Lambda(y, v, s)|\, \mathcal{N}(0, tQ) dy = 0. \tag{3.3.28}$$

Let L be a countable dense set in K. For any $y \in H, \; s \in (0,1]$ the map $\Lambda(y, \cdot, s) \in \mathcal{C}_b(H, H)$ and so the following property holds:

$$\sup_{v \in K} |\Lambda(y, v, s)| = \sup_{v \in L} |\Lambda(y, v, s)|, \quad y \in H, \; s \in (0,1].$$

We point out that for any fixed $s \in (0,1]$, the map $\sup_{v \in L} |\Lambda(\cdot, v, s)|$ is a real Borel map on H. Further we have:

$$\sup_{s \in (0,1]} \sup_{y \in H} \sup_{v \in L} |\Lambda(y, v, s)| \leq 2C\, \|\hat{D}^2 f\|_0, \tag{3.3.29}$$

where we have choosen C such that for any $v \in K$, $|v| \leq C$. Using the estimate (3.3.29) and applying the Dominated Convergence Theorem we get (3.3.28) and so (3.3.27). Using (3.3.27) it is simple to verify that for any $v \in H$, $\hat{D}^2 O_t f(\cdot)(v) \in \mathcal{C}_b(H, H)$. This way we get that $O_t f \in \mathcal{C}_s^2(H)$.

By (3.3.27) we easily can obtain that

$$\sup_{x\in H} \|\hat{D}^2 O_t f(x)\|_{\mathcal{L}(H)} \;\le\; \|\hat{D}^2 f\|_{0,\mathcal{L}(H)}.$$

Finally there results

$$\begin{aligned}\sup_{x\in H} \|D_Q^2 O_t f(x)\|_{\mathcal{L}_1(H)} \;&=\; \sup_{x\in H} \|Q^{1/2}\,\hat{D}^2 O_t f(x)\,Q^{1/2}\|_{\mathcal{L}_1(H)}\\ &\le\; \|Q^{1/2}\|_2^2\;\|\hat{D}^2 O_t f(x)\|_{0,\mathcal{L}(H)} \;\le\; \mathrm{Tr}\,(Q)\;\|\hat{D}^2 f\|_{0,\mathcal{L}(H)}.\end{aligned}$$

(b) Take $f \in D(\mathcal{A}_1)\cap \mathcal{C}_s^2(H)$. Combining (a) and Proposition 3.3.1, we easily find that $O_t f \in D(\mathcal{A}_1)\cap \mathcal{C}_s^2(H)$, $t>0$.

(c) By the Hille-Yosida Theorem we have that for any $\lambda>0$, $x\in H$,

$$R(\lambda,\mathcal{A})f(x) \;=\; \int_0^\infty e^{-\lambda t} O_t f(x)\,dt.$$

Differentiating under the integral sign, using the estimate in (a) and proceeding as for (3.3.27), it is straightforward to obtain the assertion. The proof is complete. ∎

The next theorem is a generalization of Theorem 4.1 in Cannarsa and Da Prato [12].

Theorem 3.3.5 *Let $\mathcal{A}_1$ be the linear operator defined in (3.3.1) and $\mathcal{A}$ be the generator of the heat semigroup O_t. Then following statements hold:*
(i) $D(\mathcal{A}_1)\cap \mathcal{C}_s^2(H)$ is dense in $\mathcal{C}_b(H)$;
(ii) $D(\mathcal{A}_1)\cap \mathcal{C}_s^2(H)$ is a core for $\mathcal{A}$.

Proof (i) By Theorem 2.2.7 we know that $\mathcal{C}_s^2(H)$ is dense in $\mathcal{C}_b(H)$. Thus for any $g\in \mathcal{C}_b(H)$, $\epsilon>0$, there exists $l\in \mathcal{C}_s^2(H)$ such that $\|g-l\|_0 \le \epsilon$.

By Proposition 3.3.4 we deduce that $O_t l \in D(\mathcal{A}_1)\cap \mathcal{C}_s^2(H)$, $t>0$. Now the inequality

$$\|g - O_t l\|_0 \;\le\; \|g - l\|_0 \;+\; \|l - O_t l\|_0$$

allows us to obtain the assertion, by using that O_t is strongly continuous in $\mathcal{C}_b(H)$.

(ii) We have to show that $Y = D(\mathcal{A}_1)\cap \mathcal{C}_s^2(H) \subset D(\mathcal{A})$ and moreover that Y is dense in $D(\mathcal{A})$ with respect to the graph norm.

By Theorem 3.3.2 we deduce in particular that $Y\subset D(\mathcal{A})$. Moreover by Proposition 3.3.4, we know that Y is invariant for O_t. Using also that Y is dense in $\mathcal{C}_b(H)$, see (i), we derive that Y is a core invoking a well known result, see Theorem 1.9 in Davies [25]. This concludes the proof. ∎

3.4 Interpolation spaces associated with O_t

In this section we present some interpolation results related to O_t. These results will be used in Chapter 4 and 5 in the study of second order of elliptic equations.

The following basic result is proved in Cannarsa and Da Prato [12].

Theorem 3.4.1 *Let $\theta \in (0,1)$. Then*

$$(\mathcal{C}_b(H), \mathcal{C}^1_Q(H))_{\theta,\infty} \ = \ \mathcal{C}^\theta_Q(H).$$

Now we deal with $\mathcal{D}_\mathcal{A}(\theta,\infty) = (\mathcal{C}_b(H),\, D(\mathcal{A}))_{\theta,\infty}$, $\theta \in (0,1)$, where $D(\mathcal{A})$ is endowed with the graph norm (see §1.4 for more details).

The first assertion of the next result is proved in Lemma 5.1 and Corollary 5.1 of Cannarsa and Da Prato [12].

Proposition 3.4.2 *Let $\mathcal{A}$ be the generator of O_t. Then, for any $\theta \in (0,1)$, it holds (with continuous embeddings):*

(i) $\mathcal{D}_\mathcal{A}(\theta/2,\infty) \subset \mathcal{C}^\theta_Q(H)$;

(ii) $\mathcal{C}^\theta_b(H) \subset \mathcal{D}_\mathcal{A}(\theta/2,\infty)$.

Proof **(i)** We use the following fact

$$D(\mathcal{A}) \subset \mathcal{C}^1_Q(H) \subset \mathcal{C}_b(H), \text{ with continuous embeddings .} \tag{3.4.1}$$

To prove that $D(\mathcal{A}) \subset \mathcal{C}^1_Q(H)$, we fix $u \in D(\mathcal{A})$ and set $f = \lambda u - \mathcal{A}u$, with $\lambda > 0$.

In view of the Hille-Yosida Theorem we know that

$$u = \int_0^\infty e^{-\lambda t} O_t f \; dt. \tag{3.4.2}$$

By using Proposition 3.2.2, we know in particular that $O_t f \in \mathcal{C}^1_Q(H)$. Thanks to the second estimate in (3.2.6), we can differentiate under the integral sign in (3.4.2) in order to obtain that $u \in \mathcal{C}^1_Q(H)$. Moreover there results

$$\begin{aligned} \|D_Q u\|_0 = \Big\| \int_0^\infty e^{-\lambda t} D_Q O_t f \, dt \Big\|_0 &\le \|f\|_0 \int_0^\infty \frac{e^{-\lambda t}}{\sqrt{t}} \, dt \\ = \sqrt{\frac{\pi}{\lambda}}\, \|f\|_0 &\le \sqrt{\pi}(\sqrt{\lambda}\|u\|_0 + \frac{\|\mathcal{A}u\|_0}{\sqrt{\lambda}}). \end{aligned}$$

Now letting $\lambda = 1$ in the last term we immediately find that $D(\mathcal{A}) \subset \mathcal{C}^1_Q(H)$. Moreover if u, $\mathcal{A}u \neq 0$, then, setting $\lambda = \frac{\|\mathcal{A}u\|_0}{\|u\|_0}$ in the last term, we find the following useful interpolatory estimate

$$\|D_Q u\|_0 \ \le \ 2\sqrt{\pi}\, \|u\|_0^{1/2}\, \|\mathcal{A}u\|_0^{1/2}. \tag{3.4.3}$$

By using (3.4.1) and (3.4.3), we can apply the Reiteration Theorem (see §1.4) and obtain assertion (i).

(ii) Let $g \in \mathcal{C}^\theta_Q(H)$, then for any $x \in H$, $t \ge 0$, we have

$$|O_t g(x) - g(x)| \le \int_H |g(x+\sqrt{t}y) - g(x)|\,\mathcal{N}(0,Q)dy$$

$$\le t^{\theta/2}\,[g]_\theta \int_H |y|^\theta\,\mathcal{N}(0,Q)dy = C_\theta [g]_\theta\, t^{\theta/2}.$$

From this estimate, recalling (1.4.2), assertion (ii) follows. The proof is complete. ■

The next theorem, proved in Priola and Zambotti [70], shows that actually the inclusion (i) of the previous proposition is strict.

Theorem 3.4.3 *For any* $\theta \in (0,1)$, *it holds:*

$$\mathcal{D}_{\mathcal{A}}(\theta/2,\infty) \ne \mathcal{C}^\theta_Q(H).$$

Proof Assume, by contraddiction, that there exists a $\hat\theta \in (0,1)$ such that

$$\mathcal{D}_{\mathcal{A}}(\hat\theta/2,\infty) = \mathcal{C}^{\hat\theta}_Q(H). \tag{3.4.4}$$

By (i) of Proposition 3.4.2 and (3.4.4), applying the Open Mapping Theorem, we obtain that the norms $\|\cdot\|_{\hat\theta/2,\mathcal{A}}$ and $\|\cdot\|_{\hat\theta,Q}$ are equivalent.

Now we will use the following recent result, proved in Van Neerven and Zabczyk [82] (see Theorem 3.2.1),

$$\|O_{t+h} - O_t\|_{\mathcal{L}(\mathcal{C}_b(H))} = 2, \;\; t \ge 0,\, h > 0. \tag{3.4.5}$$

Fix any $t > 0$. By (3.4.5), for any $h > 0$, there exists a map $f_h \in \mathcal{C}_b(H)$ such that $\|f_h\|_{\mathcal{C}_b(H)} \le 1$ and moreover

$$2 - h < \|O_{t+h} f_h - O_t f_h\|_0 = \|O_h O_t f_h - O_t f_h\|_0 \le [O_t f_h]_{\hat\theta/2,\mathcal{A}}\, h^{\hat\theta/2}. \tag{3.4.6}$$

Therefore once we have proved that

$$\sup_{h>0}\, [O_t f_h]_{\hat\theta/2,\mathcal{A}} < \infty, \tag{3.4.7}$$

we will obtain a contradiction, letting $h \to 0^+$ in (3.4.6). Now we check (3.4.7). Using the fact that $\|\cdot\|_{\hat\theta/2,\mathcal{A}}$ is equivalent to $\|\cdot\|_{\hat\theta,Q}$ and formula 3.2.6, we infer

$$\begin{aligned}\|O_t f_h\|_{\hat\theta/2,\mathcal{A}} &\le C_1\,\|O_t f_h\|_{\hat\theta,Q}\\ &\le C\|O_t f_h\|_{1,Q} \le \frac{C}{\sqrt{t}}\|f_h\|_0 \le \frac{C}{\sqrt{t}}, \quad h > 0.\end{aligned} \tag{3.4.8}$$

Thus (3.4.7) is verified and the assertion follows. The proof is complete. ■

By (ii) of Proposition 3.4.2 and by Theorem 3.4.3, we deduce the following useful statement.

Corollary 3.4.4 *For any* $\theta \in (0,1)$, *it holds:*

$$\mathcal{C}^\theta_b(H) \ne \mathcal{C}^\theta_Q(H).$$

Part II

Elliptic equations in infinite dimensions

Chapter 4

Schauder estimates for second order elliptic operators in $\mathcal{C}_b(H)$

In this chapter we prove a sharp form of Schauder estimates for a second order infinite-dimensional elliptic operator with Hölder continuous coefficients taking values in the space of Hilbert-Schmidt type operators.

4.1 Introduction

In this chapter we are concerned with the infinite dimensional elliptic equation

$$\lambda u(x) - \frac{1}{2}\mathrm{Tr}[Q(x)D^2u(x)] = f(x), \quad x \in H, \quad \lambda > 0, \tag{4.1.1}$$

where H is a real separable Hilbert space and f, $u : H \mapsto \mathbb{R}$ belong to $C_b(H)$, the space of all real bounded uniformly continuous functions. Here we mainly present results contained in Priola and Zambotti [70].

An important motivation to study equation (4.1.1) comes from a well known connection with stochastic differential equations as

$$dX(t) = Q^{1/2}(X(t))\, dW(t). \tag{4.1.2}$$

Equations like (4.1.2) can be treated by usual techniques if $Q(x)$ is Lipschitz continuous with respect to x, see for instance Da Prato and Zabczyk [23], [24]. However solving directly the deterministic equation (4.1.1) allows to establish existence and uniqueness in law for solutions of (4.1.2), also when the coefficients are only Hölder continuous (we refer to Zambotti [86], [87] for details).

Later Piech (see Piech [63]) has constructed a foundamental solution for (4.1.1) in case of

$$Q(x) = Q^{1/2}(I + F(x))Q^{1/2}, \quad x \in H, \tag{4.1.3}$$

where $F(x)$ is a family of trace-class operators, satisfying strong smoothness assumptions.

In Cannarsa and Da Prato [12], [13], the equation (4.1.1) has been studied when F is Hölder-continuous from H with values in the space $\mathcal{L}_1(H)$ of all trace class operators. It is shown that when $f \in C_Q^\theta(H)$ (the set of all functions that are $\theta-$Hölder continuous in the directions of $Q^{1/2}H$, $\theta \in]0,1[$), the solution u of (4.1.1) belongs to $\mathcal{C}_Q^2(H)$ (see Chapter 1 for a precise definition) and its second Q-derivative, $D_Q^2 u$, is a Q-Hölder continuous map with values in the space $\mathcal{L}(H)$ of all bounded linear operators in H. However they give no informations about a typical regularity problem arising in infinite dimensions: whether, for a solution u of (4.1.1), the bounded linear operator $D_Q^2 u(x)$, $x \in H$, is compact, or of Hilbert-Schmidt type, or of trace class, etc. Because of this lack, in Cannarsa and Da Prato [12], very restrictive hypotheses on F are required.

In this chapter we prove that $D_Q^2 u(x)$ is in fact of Hilbert-Schmidt type. Note that in light of the Gross results (see Gross [41]), this seems to be the best possible regularity result for $D_Q^2 u(x)$ even when $F = 0$. Using this result we are able to relax the hypotheses on the coefficients F of (4.1.1), proving again existence and uniqueness for solutions.

Another important phenomenon, typical of the infinite dimensions, is the difficulty of characterizing the domain of the generator $\mathcal{A}$ of the heat semigroup in $\mathcal{C}_b(H)$ and its interpolation spaces $(C_b(H), D(\mathcal{A}))_{\theta/2,\infty}$. This problem arises in the study of the spatial regularity for solutions of elliptic equations like (4.1.1). When $H = \mathbb{R}^n$, it is well known that the following interpolatory result holds

$$(C_b(\mathbb{R}^n), D(\mathcal{A}))_{\theta/2,\infty} = C_b^\theta(\mathbb{R}^n), \tag{4.1.4}$$

for $\theta \in]0,1[$. We stress that (4.1.4) is a key step in the modern treatment of Schauder estimates for (4.1.1) (see for instance Lunardi [55] and Triebel [78]).

On the contrary, in infinite dimensions, we only have the *strict* inclusion, see Theorem 3.4.3,

$$(C_b(H), D(\mathcal{A}))_{\theta/2,\infty} \subset C_Q^\theta(H) \tag{4.1.5}$$

and is a long standing problem the characterization of $(C_b(H), D(\mathcal{A}))_{\theta/2,\infty}$.

In the case of the equation with constant coefficients

$$\lambda\psi(x) - \frac{1}{2}\mathrm{Tr}[QD^2\psi(x)] = f(x), \quad x \in H, \quad \lambda > 0, \tag{4.1.6}$$

we prove that, for all $N \in \mathcal{L}_2(H)$, $\quad \mathrm{Tr}[ND_Q^2\psi] \in (C_b(H), D(\mathcal{A}))_{\theta/2,\infty}$ $\quad$ and

$$\sup_{\|N\|_{\mathcal{L}_2} \le 1} \|\mathrm{Tr}[ND_Q^2\psi]\|_{(C_b(H),D(\mathcal{A}))_{\theta/2,\infty}} \le C\|f\|_{\theta,Q}, \tag{4.1.7}$$

where $C = C(\lambda, \theta, Q)$. It is a deep fact that this *sharp* form of Schauder estimates allows to obtain for the general equation (4.1.1) that $\quad \mathrm{Tr}[ND_Q^2 u] \in C_Q^\theta(H)$ $\quad$ and

$$\sup_{\|N\|_{\mathcal{L}_2} \le 1} \|\mathrm{Tr}[ND_Q^2 u]\|_{C_Q^\theta(H)} \le C\|f\|_{\theta,Q} \tag{4.1.8}$$

which is weaker than (4.1.7) but nonetheless sufficient in order to prove existence of solutions for (4.1.1). It seems that our considerations are a new and consistent contribution to the difficult problem to study regularity of domains of differential operators in infinite dimensions.

In this Chapter there are two main results. In Section 4.2 we prove the first one that concerns the optimal regularity (see (4.1.7)) for solutions of equation (4.1.6). To this purpose we only use analytic tools: estimates on the heat semigroup (see Proposition 3.2.3) and Interpolation Theory (as in Cannarsa and Da Prato [12] and Da Prato and Lunardi [21]).

Using this result, in Section 4.3, we are able to treat equations (4.1.1) when F is only a Q-Hölder-continuous map with values in the space $\mathcal{L}_2(H)$ of Hilbert-Schmidt operators in H. In Theorem 4.3.6 we prove the a priori estimates (4.1.8) for solutions of (4.1.1). The proof of this result requires a new method and relies on a non standard interpolation lemma of independent interest (see Lemma 4.3.7), involving Hilbert-Schmidt norms of second derivatives of mappings.

We prove also a maximum principle for (4.1.1) that extends Theorem A1 of Cannarsa and Da Prato [12], see Theorem 4.3.3. The proof is simpler than that of Theorem A1 thanks to Lemma 4.3.5. Finally, by adapting the classical continuity method to equation (4.3.1), we obtain a theorem of existence, uniqueness and optimal regularity for solutions u of (4.1.1) (see Theorem 4.3.9).

4.2 Optimal regularity results: constant coefficients

We recall some notations, refering to Chapter 1 and Section 3.2 for more details. Let H be a real separable Hilbert space with inner product $< \cdot, \cdot >$ and norm $|\cdot|$. $\mathcal{L}(H)$ stands for the Banach space of all bounded linear operators on H, endowed with the operator norm.

Let Q be a positive self-adjoint trace class operator in H, we fix once and for all an orthonormal basis of H, $\{e_k\}_{k\geq 1}$, that diagonalizes Q: for any $x \in H$, $Qx = \sum_{k=1}^{\infty} \lambda_k x_k e_k$ with $x_k =< x, e_k >$, $\lambda_k > 0$, $k \geq 1$.

Moreover $\mathcal{N}(x, tQ)$ denotes the Gaussian measure in H with mean $x \in H$ and covariance operator tQ.

Let $\mathcal{B}$ be a closed operator on a Banach space X. For any $\lambda \in \rho(\mathcal{B})$, $\rho(\mathcal{B})$ stands for the resolvent set of $\mathcal{B}$, we can consider the *resolvent operator* of $\mathcal{B}$, defined in the following way:

$$R(\lambda, \mathcal{B}) \stackrel{\text{def}}{=} (\lambda I - \mathcal{B})^{-1}. \tag{4.2.1}$$

In this section we are dealing with the following equation

$$\lambda u(x) - \frac{1}{2}\text{Tr}\,[QD^2u(x)] = f(x), \quad x \in H, \ \ \lambda > 0.$$

Considering the orthonormal basis $\{e_k\}_{k\geq 1}$ of H, previously fixed, the equation becomes

$$\lambda u(x) - \frac{1}{2}\text{Tr}\,[QD^2u(x)] = \lambda u(x) - \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k D_{kk} u(x), \ \ x \in H, \tag{4.2.2}$$

where $D_k u$ is the partial derivative of u in the direction e_k, $k \geq 1$ and we set $D_{hk}u = D_h(D_k u)$, $h, k \geq 1$. We write (4.2.2) as

$$\lambda u - \mathcal{A}u = f, \tag{4.2.3}$$

where $\mathcal{A}$ is the generator of the heat semigroup O_t on $\mathcal{C}_b(H)$ (see (3.2.3)).

We recall the following result about Hilbert-Schmidt operators, that will be frequently used (see Lemma 1.1.3):

Denote by $\mathcal{F}_1$ the subset of $\mathcal{L}(H)$ of all finite rank operators N, such that $\|N\|_{\mathcal{L}_2(H)} \leq 1$. Let $L \in \mathcal{L}(H)$, then $L \in \mathcal{L}_2(H)$ if and only if

$$\sup_{N \in \mathcal{F}_1} |\mathrm{Tr}\,(NL)| = c < \infty. \tag{4.2.4}$$

Moreover if (4.2.4) holds then $\|L\|_2 = c$.
Now we prove a preliminary non optimal regularity result for (4.2.3).

Proposition 4.2.1 *Consider $u = R(\lambda, \mathcal{A})f$, $f \in \mathcal{C}_Q^\theta(H)$, $\lambda > 0$, $\theta \in (0,1)$. Then $u \in \mathcal{C}_Q^2(H)$ and $D_Q^2 u \in \mathcal{C}_b(H, \mathcal{L}_2(H))$. Moreover there exists a constant $c = c(\lambda, Q, \theta) > 0$, such that:*

$$\|u\|_{2,Q} + \|D_Q^2 u\|_{0,\mathcal{L}_2(H)} \leq c\, \|f\|_{\theta,Q}. \tag{4.2.5}$$

Proof We have, by the Hille-Yosida Theorem,

$$u = \int_0^\infty e^{-\lambda t}\, O_t f\, dt, \tag{4.2.6}$$

where the integral has to be interpreted in the Bochner sense with values in $\mathcal{C}_b(H)$. By the first estimate of (3.2.6), differentiating under the integral sign in (4.2.6) and taking into account that

$$\begin{aligned} &|< D_Q O_t f(x) - D_Q O_t f(z), v >|^2 \leq \frac{1}{t}\omega_f(|x-z|)^2 \int_H |< (tQ)^{-1/2}y, v >|^2 \mathcal{N}(0,tQ)dy \\ &\leq \frac{1}{t}\,\omega_f(|x-z|)^2\,|v|^2, \quad x,\ z,\ v \in H,\ t > 0, \end{aligned} \tag{4.2.7}$$

we deduce easily that $u \in \mathcal{C}_Q^1(H)$. To get more regularity for u, consider that, from Proposition 3.2.3, there results, for any $t > 0$,

$$\|D_Q^2 O_t h\|_{0,\mathcal{L}_2(H)} \leq \frac{2}{t}\|h\|_0, \quad \|D_Q^2 O_t g\|_{0,\mathcal{L}_2(H)} \leq \frac{1}{\sqrt{t}}\|g\|_{1,Q}, \quad h \in \mathcal{C}_b(H),\ g \in \mathcal{C}_Q^1(H).$$

By Theorem 1.4.1, interpolating between these estimates, since $f \in (\mathcal{C}_b(H), \mathcal{C}_Q^1(H))_{\theta,\infty}$ (see Theorem 3.4.1), one has

$$\|D_Q^2 O_t f\|_{0,\mathcal{L}_2(H)} \leq c_\theta\, t^{\theta/2-1}\, \|f\|_{\theta,Q}, \quad t > 0. \tag{4.2.8}$$

Using this estimate, we can readily derive that there exists $D_Q^2u(x) \in \mathcal{L}(H)$ for any $x \in H$ and

$$< D_Q^2u(x)(w), v >= \int_0^\infty e^{-\lambda t} \ < D_Q^2 O_t f(x)(w), v > \ dt, \ \ x, w, v \in H.$$

To get that $D_Q^2u(x) \in \mathcal{L}_2(H)$, we use formula (4.2.4). Let $N \in \mathcal{F}_1$, in $N(H)$ we can choose an orthonormal basis $(l_k), k = 1, ..n$. Then there results

$$\begin{aligned} |\mathrm{Tr}\ (N D_Q^2 u(x))| \ &= \Big| \sum_{k=1}^n < D_Q^2 u(x)(l_k), N^* l_k > \Big| \\ &\leq \int_0^\infty e^{-\lambda t}\, |\mathrm{Tr}\ (N D_Q^2 O_t f(x))|\, dt \\ &\leq c_\theta \|f\|_{\theta,Q} \int_0^\infty e^{-\lambda t}\, t^{\theta/2-1}\, dt \ = \ C_{\theta,\lambda} \|f\|_{\theta,Q}, \end{aligned} \tag{4.2.9}$$

so that $D_Q^2u(x) \in \mathcal{L}_2(H)$, $x \in H$, and moreover $\|D_Q^2 u\|_{0,\mathcal{L}_2(H)} \leq C_{\theta,\lambda} \ \|f\|_{\theta,Q}$.

It remains to establish the uniform continuity of D_Q^2u. This is equivalent to show that for any sequence $(z_n) \subset H$ such that $z_n \to 0$ as $n \to \infty$, it holds:

$$\lim_{n\to\infty} \sup_{x\in H} \|D_Q^2 u(x+z_n) - D_Q^2 u(x)\|_{\mathcal{L}_2(H)} \ = 0. \tag{4.2.10}$$

Let us fix a countable dense subset L of H. Since $\mathcal{L}_2(H)$ is separable, we can choose a countable dense subset $\mathcal{M}$ of $\mathcal{F}_1$. Now using that for any $A \in \mathcal{L}_2(H)$ the linear map: $\mathcal{L}_2(H) \to \mathbb{R}$, $\ N \mapsto \mathrm{Tr}\ (NA)$ is continuous, we obtain

$$\|T\|_2 \ = \ \sup_{N\in\mathcal{F}_1} |\mathrm{Tr}\ (NT)| \ = \ \sup_{N\in\mathcal{M}} |\mathrm{Tr}\ (NT)|, \quad T \in \mathcal{L}_2(H). \tag{4.2.11}$$

From this formula it follows readily that the maps $\gamma_n :]0, \infty[\to \mathbb{R}$,

$$\begin{aligned} \gamma_n(t) \ &= \ \sup_{x\in H} \|D_Q^2 O_t f(x+z_n) - D_Q^2 O_t f(x)\|_{\mathcal{L}_2(H)} \\ &= \ \sup_{x\in L,\, N\in\mathcal{M}} |\mathrm{Tr}\ (N[D_Q^2 O_t f(x+z_n) - D_Q^2 O_t f(x)])|, \ \ t > 0, \end{aligned}$$

are Borel for any $n \geq 1$. Thus we can write

$$\sup_{x\in H} \|D_Q^2 u(x+z_n) - D_Q^2 u(x)\|_{\mathcal{L}_2(H)} \ \leq \int_0^\infty e^{-\lambda t}\, \gamma_n(t)\, dt.$$

Now $\lim_{n\to\infty} \gamma_n(t) = 0$, $t > 0$, by Proposition 3.2.3. Hence letting $n \to \infty$ in right-hand side of the last formula, we find (4.2.10) by the Dominated Convergence Theorem. This completes the proof. ∎

In the next result we present Schauder estimates for (4.2.3) and improve Theorem 5.1 of Cannarsa and Da Prato [12]. To this purpose we will use results concerning real interpolation spaces related to O_t, see Section 3.4. We recall that

$$\mathcal{D}_{\mathcal{A}}(\theta/2, \infty) \overset{\mathrm{def}}{=} (\mathcal{C}_b(H), D(\mathcal{A}))_{\theta/2,\infty}, \quad \theta \in (0,1).$$

Theorem 4.2.2 *Consider* $u = R(\lambda,\mathcal{A})f$, $f \in \mathcal{C}^\theta_Q(H)$ $\lambda > 0$, $\theta \in (0,1)$. *Then* $u \in \mathcal{C}^2_Q(H)$ *and* $D^2_Q u \in \mathcal{C}_b(H,\mathcal{L}_2(H))$. *Moreover for any* $N \in \mathcal{F}_1$ *(see (4.2.4)) one has that* $\mathrm{Tr}\,(ND^2_Q u) \in \mathcal{D}_{\mathcal{A}}(\theta/2,\infty)$ *and there exists a constant* $c = c(\lambda,Q,\theta) > 0$*, such that:*

$$\|u\|_{2,Q} + \sup_{N\in\mathcal{F}_1} \|\mathrm{Tr}(ND^2_Q u)\|_{\theta/2,\mathcal{A}} \ \le c\,\|f\|_{\theta,Q}. \tag{4.2.12}$$

In particular (4.2.12) implies that $D^2_Q u \in C^\theta_Q(H,\mathcal{L}_2(H))$ *and it holds:*

$$\lambda\|u\|_{\theta,Q} + \|\mathcal{A}u\|_{\theta,Q} + \|u\|_{2,Q} + \|D^2_Q u\|_{\theta,Q,\mathcal{L}_2} \ \le \hat{c}\,\|f\|_{\theta,Q}. \tag{4.2.13}$$

Proof First we verify that $\lambda\|u\|_{\theta,Q} + \|\mathcal{A}u\|_{\theta,Q} \le 3\|f\|_{\theta,Q}$.
This fact follows, since $\|O_t f\|_{\theta,Q} \le \|f\|_{\theta,Q}$, $t \ge 0$, and so

$$\|u\|_{\theta,Q} \le \int_0^\infty e^{-\lambda s}\,\|f\|_{\theta,Q}\,ds = \frac{1}{\lambda}\|f\|_{\theta,Q}.$$

Moreover $\|\mathcal{A}u\|_{\theta,Q} \le \lambda\|u\|_{\theta,Q} + \|f\|_{\theta,Q} \le 2\|f\|_{\theta,Q}$. In order to prove (4.2.12), let $N \in \mathcal{F}_1$; we show that $\mathrm{Tr}\,(ND^2_Q u) \in D_{\mathcal{A}}(\theta/2,\infty)$. For any function $h \in \mathcal{C}^2_Q(H)$, we set:

$$Uh(x) \ = \ \mathrm{Tr}\,(ND^2_Q h)(x), \ \ x \in H.$$

Thus for any for $\xi \in [0,1]$, we have to estimate $I_\xi \ = \ \sup_{x\in H} \ |O_\xi Uu(x) - Uu(x)|$, $\xi \in [0,1]$. Remark that it holds

$$O_t Uh(x) \ = \ UO_t h(x), \quad h \in \mathcal{C}^2_Q(H), \ \ x \in H, \ t \ge 0. \tag{4.2.14}$$

Indeed in $N(H)$ we can choose an orthonormal basis $(l_k), k = 1,..n$. Then we have, by the Dominated Convergence Theorem,

$$\begin{aligned}
\mathrm{Tr}\,(ND^2_Q O_t h)(x) &= \sum_{k=1}^n \int_H < D^2_Q h(x+y)(l_k), N^* l_k > \ \mathcal{N}(0,tQ)dy \\
&= \int_H \mathrm{Tr}\,[ND^2_Q h(x+y)]\,\mathcal{N}(0,tQ)dy \ = O_t[\mathrm{Tr}\,(ND^2_Q h)](x),
\end{aligned}$$

where $x \in H$. Formula (4.2.14) yields, applying (4.2.8),

$$\begin{aligned}
I_\xi \ &= \ \sup_{x\in H} \Big| \int_0^\infty e^{-\lambda t} \Big(UO_{t+\xi} f(x) - UO_t f(x) \Big)\,dt \Big| \\
&= \ \sup_{x\in H} \Big| (e^{\lambda\xi} - 1) \int_0^\infty e^{-\lambda t}\,UO_t f(x) dt \ - \ e^{\lambda\xi} \int_0^\xi e^{-\lambda t} UO_t f(x) dt \Big| \\
&\le \ c\,\|f\|_{\theta,Q} \Big[(e^{\lambda\xi} - 1) \int_0^\infty e^{-\lambda t} t^{\theta/2-1} dt \ + \ e^{\lambda\xi} \int_0^\xi e^{-\lambda t} t^{\theta/2-1} dt \Big] \\
&\le \ \hat{C}\,\|f\|_{\theta,Q}\,\xi^{\theta/2}, \qquad\qquad \xi \in [0,1],
\end{aligned} \tag{4.2.15}$$

where $\hat{C} = \hat{C}(\lambda, Q, \theta)$.

Hence by (4.2.15) and (1.4.2) we obtain that

$$\text{Tr }(ND_Q^2 u) \in D_{\mathcal{A}}(\theta/2, \infty) \subset \mathcal{C}_Q^{\theta}(H)$$

and

$$\|\text{Tr }(ND_Q^2 u)\|_{\mathcal{C}_Q^{\theta}(H)} \leq C_1 \|\text{Tr }(ND_Q^2 u)\|_{D_{\mathcal{A}}(\theta/2,\infty)} \leq C_2 \|f\|_{\theta,Q}$$

where C_1 and C_2 do not depend on N. Then, taking the supremum over $N \in \mathcal{F}_1$, we infer

$$\|D_Q^2 u\|_{\theta,Q,\mathcal{L}_2} \leq 2C_1 \sup_{N\in\mathcal{F}_1} \|\text{Tr}(ND_Q^2 u)\|_{\theta/2,\mathcal{A}} \leq 2C_2 \|f\|_{\theta,Q}. \tag{4.2.16}$$

Combining (4.2.16) with Proposition 4.2.1, the thesis follows. ∎

Now we extend Theorem 4.2.2 to elliptic equations of the form

$$\lambda u(x) - \frac{1}{2}\text{Tr}[Q^{1/2}(I + F)Q^{1/2}D^2 u(x)] = f(x), \quad x \in H, \lambda > 0. \tag{4.2.17}$$

where

$$\text{F is symmetric, non negative and belongs to } \mathcal{L}_2(H). \tag{4.2.18}$$

We can introduce the operator $S = Q^{1/2}(1 + F)Q^{1/2}$. It is easy to show that S is a positive trace class operator on H. Hence we can define the heat semigroup O_t^S on $\mathcal{C}_b(H)$, associated with S,

$$O_t^S f(x) = \int_H f(x + y)\, \mathcal{N}(0, tS) dy,\ f \in \mathcal{C}_b(H),\ x \in H.$$

Denoting by $\mathcal{A}^S$, the generator of O_t^S, equation (4.2.17) can be rewritten as

$$\lambda u(x) - \mathcal{A}^S u(x) = f(x), \quad x \in H, \lambda > 0. \tag{4.2.19}$$

In order to study (4.2.19), we need the following result, proved in in Cannarsa and Da Prato [12].

Proposition 4.2.3 *Let $S = Q^{1/2}(1 + F)Q^{1/2}$, where F satisfies (4.2.18). Then $S^{1/2}(H) = Q^{1/2}(H)$ and the linear operators $S^{1/2}Q^{-1/2}$, $Q^{1/2}S^{-1/2}$, $Q^{-1/2}S^{1/2}$, $S^{-1/2}Q^{1/2}$ are bounded.*

Proof We remark that, for any $x \in H$,

$$< Sx, x >=< (I + F)Q^{1/2}x, Q^{1/2}x >=< Q^{1/2}x, Q^{1/2}x > + < FQ^{1/2}x, Q^{1/2}x > .$$

Since F is non negative, we infer

$$|Q^{1/2}x|^2 \leq |S^{1/2}x|^2 \leq \|I + F\|_{\mathcal{L}(H)} |Q^{1/2}x|^2, \ x \in H. \tag{4.2.20}$$

We only prove $S^{1/2}Q^{-1/2}$ and $Q^{-1/2}S^{1/2}$ are bounded and that $S^{1/2}(H) \subset Q^{1/2}(H)$. The reminder assertions can be proved similarly.

We set $L = S^{1/2}Q^{-1/2}$. By the last estimate, it follows that L can be extended to a bounded linear operator on H. We again denote by L this extension, that is unique. It is easy to check that $LQ^{1/2} = S^{1/2}$. Moreover, by taking the adjoints, we find

$$Q^{1/2}L^* = S^{1/2}.$$

This implies that $S^{1/2}(H) \subset Q^{1/2}(H)$. Thus, applying the Closed Graph Theorem, we deduce that the linear operator $Q^{-1/2}S^{1/2}$ is bounded. The proof is complete. ■

Corollary 4.2.4 *Let $S = Q^{1/2}(1+F)Q^{1/2}$, where F satisfies (4.2.18). Then it holds:*

$$\mathcal{C}^\theta_S(H) = \mathcal{C}^\theta_Q(H), \ \ \theta \in (0,1), \qquad \mathcal{C}^n_Q(H) = \mathcal{C}^n_S(H), \ \ n \geq 1.$$

Proof The proof is simple. We only verify that $\mathcal{C}^1_S(H) \subset \mathcal{C}^1_Q(H)$. To this purpose let $f \in \mathcal{C}^1_S(H)$. There results, applying the previous proposition,

$$\lim_{s\to 0^+} \frac{f(x+sQ^{1/2}v)}{s} = \lim_{s\to 0^+} \frac{f(x+sS^{1/2}S^{-1/2}Q^{1/2}v)}{s}$$

$$=< D_Sf(x), S^{-1/2}Q^{1/2}v >=< (S^{-1/2}Q^{1/2})^*D_Sf(x), v >=< Q^{1/2}S^{-1/2}D_Sf(x), v >,$$

where x, $v \in H$. This implies that $f \in \mathcal{C}^1_Q(H)$ and $D_Qf = Q^{1/2}S^{-1/2}D_Sf$. Similarly one checks the other statements. ■

The next result generalizes Theorem 5.2 in Cannarsa and Da Prato [12].

Theorem 4.2.5 *Let $f \in C^\theta_Q(H)$ and $u = R(\lambda, \mathcal{A}^S)$, where $S = Q^{1/2}(1+F)Q^{1/2}$, and F satisfies (4.2.18). Then $u \in C^2_Q(H)$, $D^2_Qu \in \mathcal{C}^\theta_Q(H, \mathcal{L}_2(H))$ and there exists a constant $C > 0$, which does not depend on f, such that*

$$\|u\|_{2,Q} + \|D^2_Qu\|_{\theta,Q,\mathcal{L}_2} \leq C\, \|f\|_{\theta,Q}. \tag{4.2.21}$$

Proof By Corollary 4.2.4 we know that $f \in \mathcal{C}^\theta_S(H)$. Moreover, by Theorem 4.2.2, it follows that $u \in C^2_S(H)$, $D^2_Su \in \mathcal{C}^\theta_S(H, \mathcal{L}_2(H))$ and there exists a constant $c > 0$ such that

$$\|u\|_{2,S} + \|D^2_Su\|_{\theta,Q,\mathcal{L}_2} \leq C\, \|f\|_{\theta,Q}. \tag{4.2.22}$$

Combining Corolllary 4.2.4 and formula (4.2.22), we easily prove the assertion and find with a constant C, that only depends on λ, θ, Q, F. ■

4.3 Elliptic equations with variable coefficients

We consider now the following elliptic equation

$$\lambda u(x) - \mathcal{A}u(x) - \frac{1}{2}\mathrm{Tr}\,(F(x)D_Q^2 u(x)) = f(x), \ \ x \in H, \, \lambda > 0, \tag{4.3.1}$$

where $f \in \mathcal{C}_Q^\theta(H)$, $\theta \in (0,1)$ and F satisfies the following assumptions:

Hypothesis 4.3.1

(i) $F : H \to \mathcal{L}_2(H)$;

(ii) $F(x)$ *is self-adjoint and non negative,* $x \in H$;

(iii) $F \in \mathcal{C}_Q^\theta(H, \mathcal{L}_2(H))$.

We make precise the notion of solution for (4.3.1).

Definition 4.3.2 *Consider equation (4.3.1) with $f \in \mathcal{C}_b(H)$ and F fulfilling Hypothesis 4.3.1. A solution of (4.3.1) is a map $u \in D(\mathcal{A}) \cap \mathcal{C}_Q^2(H)$, such that $D_Q^2 u \in \mathcal{C}_Q^\theta(H, \mathcal{L}_2(H))$ and in addition satisfies equation (4.3.1).* ∎

In order to treat equation (4.3.1) we proceed in two steps. The next section is devoted to prove a maximum principle for (4.3.1). Note that a similar result is proved in Theorem A1 of Cannarsa and Da Prato [12]. However we will give here a simpler and direct proof.

In the second section we have to prove an a priori estimate for solutions of (4.3.1). This is achieved by means of a non standard interpolation lemma, see Lemma 4.3.7. This involves the Hilbert-Schmidt norm of the second Q-derivative of u .
Finally, by the standard continuity method, we will be able to solve the equation.

4.3.1 The maximum principle in infinite dimensions

Here we prove the following maximum principle that extends Theorem A1 of Cannarsa and Da Prato [12] in order to treat equation (4.3.1).

Theorem 4.3.3 *Consider equation (4.3.1), with $f \in \mathcal{C}_b(H)$, $\lambda > 0$ and F that fulfills Hypothesis 4.3.1. Let u be a solution, then it holds:*

$$\|u\|_0 \ \leq \ \frac{1}{\lambda}\ \|f\|_0. \tag{4.3.2}$$

The proof of the previous theorem requires the following lemma, similar to Lemma A1 in Cannarsa and Da Prato [12].

Lemma 4.3.4 *Let $u \in C_b(H)$. Then for any $\varepsilon > 0$ there exists $u_\varepsilon \in C_b(H)$ attaining a maximum in H, such that $u - u_\varepsilon \in \mathcal{C}_b^2(H)$ and for some constant $C > 0$, independent of u and ε, there results*

$$\|u - u_\varepsilon\|_{\mathcal{C}_b^2(H)} \ \leq \ C\,\varepsilon \tag{4.3.3}$$

Proof First we may assume that $u \geq 0$, without loss of generality, possibly replacing u with $u - \inf_{x \in H} u(x)$.

Now fix $\epsilon > 0$ and let $x_\varepsilon \in H$ be such that $u(x_\varepsilon) > \|u\|_0 - \varepsilon$. Let $\eta \in C_b^\infty([0, +\infty[)$ such that

$$\begin{aligned} &0 \leq \eta \leq 1,\ \eta(0) = 1,\ \eta(r) = 0\ , \forall\, r \geq 1, \\ &|\eta'(r)| \leq 2, \quad |\eta''(r)| \leq 4, \quad \forall\, r \geq 0. \end{aligned} \tag{4.3.4}$$

We define $v_\varepsilon(x) = u(x) + 2\varepsilon\, \eta\left(|x - x_\varepsilon|^2\right)$, so that $\sup\limits_{x \in H} v_\varepsilon(x) = \sup\limits_{|x - x_\varepsilon| \leq 1} v_\varepsilon(x)$.

Indeed, $v_\varepsilon(x_\varepsilon) \geq \|u\|_0 + \varepsilon$ and $v_\varepsilon(x) \leq \|u\|_0$ for $|x - x_\varepsilon| \geq 1$.

By the Asplund Theorem, see for instance Aubin [5], page 127, we know that there exists $p_\varepsilon \in H$ such that the function

$$x \mapsto v_\varepsilon(x) + < p_\varepsilon, x >$$

attains a maximum on the set $|x - x_\varepsilon| \leq 1$ and further

$$|p_\varepsilon| \leq \frac{\varepsilon}{4(2 + |x_\varepsilon|)}\ . \tag{4.3.5}$$

Let $\rho \in C_b^\infty([0, +\infty[)$ be such that

$$\begin{aligned} &0 \leq \rho \leq 1\ ;\ \rho(r) = 1\ , \forall r \leq 1\ ; \rho(r) = 0\ , \forall r \geq 2\ ; \\ &|\rho'(r)| \leq 2\ , \quad |\rho''(r)| \leq 4\ \ , \forall r \geq 0 \end{aligned} \tag{4.3.6}$$

and define

$$u_\varepsilon(x) = v_\varepsilon(x) + \ \rho\left(|x - x_\varepsilon|^2\right)\ \langle p_\varepsilon, x\rangle, \quad x \in H\ .$$

From our previous observations, it follows that u_ε attains a maximum on $|x - x_\varepsilon| \leq 1$. We claim that

$$\sup_{x \in H} u_\varepsilon(x) \ = \sup_{|x - x_\varepsilon|\ \leq 1} u_\varepsilon(x). \tag{4.3.7}$$

Indeed we have

$$\begin{aligned} &u_\varepsilon(x_\epsilon) = \ v_\varepsilon(x_\epsilon) + \ < p_\epsilon, x_\epsilon > \\ &= u(x_\epsilon) + \ 2\epsilon + \ < p_\epsilon, x_\epsilon > \geq \ \|u\|_0 \ + \epsilon - \ |p_\epsilon||x_\epsilon| \\ &\geq \ \|u\|_0 + \epsilon - \ \frac{\varepsilon |x_\epsilon|}{4(2 + |x_\varepsilon|)} \geq \ \|u\|_0 + \frac{3}{4}\epsilon \end{aligned}$$

and, on the other hand, $1 \leq |x - x_\varepsilon| \leq 2$ implies that

$$u_\varepsilon(x) \ \leq \ \|u\|_0 \ + |p_\varepsilon||x| \ \leq \ \|u\|_0 + \frac{\varepsilon}{4}\ .$$

Moreover, for $|x - x_\varepsilon| \geq 2$, we have that $u_\varepsilon(x) \leq \|u\|_0$.

Formula (4.3.3) follows by explicit computations. Note that

$$u - u_\epsilon(x) \;=\; 2\varepsilon\,\eta\left(|x - x_\varepsilon|^2\right) + \rho\left(|x - x_\varepsilon|^2\right)\langle p_\varepsilon, x\rangle, \;\; x \in H.$$

We only estimate the second derivative of $\psi(x) \stackrel{\text{def}}{=} \eta\,(|x - x_\varepsilon|^2)$.

$$\begin{aligned} D^2\psi(x) = \; & 4\eta''\left(|x - x_\varepsilon|^2\right)\;(x - x_\epsilon) \otimes (x - x_\epsilon) \\ & + \eta'\left(|x - x_\varepsilon|^2\right) I_H, \;\; x \in H, \end{aligned} \tag{4.3.8}$$

where for any $a, b \in H$, $a \otimes b(u, v) \stackrel{\text{def}}{=} < a, u >< b, v >$, $u,\ v \in H$. The previous formula yields that

$$\|D^2\psi\|_{0,\mathcal{L}(H)} \leq \; C(\|\eta''\|_0 + \;\|\eta'\|_0).$$

The proof is complete. ■

By using the space $\mathcal{C}^2_s(H)$, we define the following differential operator $\mathcal{A}_0$:

$$\begin{cases} D(\mathcal{A}_0) \;=\; \{f \in \mathcal{C}^2_s(H) \text{ such that } \; Q^{1/2}\hat{D}^2 f(\cdot)Q^{1/2} \in \mathcal{C}_b(H, \mathcal{L}_1(H))\}; \\ \\ \mathcal{A}_0 \; : D(\mathcal{A}_0) \;\to\; \mathcal{C}_b(H), \;\; \mathcal{A}_0 f(x) \stackrel{\text{def}}{=} \dfrac{1}{2}\text{Tr}\,[Q\hat{D}^2 f(x)] = \dfrac{1}{2}\text{Tr}\,[D^2_Q f(x)], \end{cases} \tag{4.3.9}$$

where $f \in D(\mathcal{A}_0)$, $x \in H$ and $\hat{D}^2 f$ denotes the second Hadamard derivative of f.

Since $\mathcal{A}_0$ is the restriction of $\mathcal{A}_1$ to $D(\mathcal{A}_1) \cap \mathcal{C}^2_s(H)$, see Chapter 3, applying Theorem 3.3.5, we know that $\mathcal{A}$ is the *closure* of $\mathcal{A}_0$.

In order to prove Theorem 4.3.3, we need the following lemma. It also allows to semplify the proof of Theorem A1 in Cannarsa and Da Prato [12].

Lemma 4.3.5 *Let $u \in D(\mathcal{A}) \cap \mathcal{C}^2_Q(H)$ and in addition suppose that $D^2_Q u \in \mathcal{C}_b(H, \mathcal{L}_2(H))$. Then there exists a sequence $(u_n) \subset D(\mathcal{A}_0)$, such that*
(i) $u_n \to u$, $\mathcal{A}_0 u_n \to \mathcal{A}u$, as $n \to \infty$, in $\mathcal{C}_b(H)$;
(ii) $u_n \to u$, as $n \to \infty$, in $\mathcal{C}^2_Q(H)$ and further

$$\lim_{n\to\infty} \|D^2_Q u_n \;-\; D^2_Q u\|_{0,\mathcal{L}_2(H)} \;=\; 0 \tag{4.3.10}$$

Proof By Theorem 3.3.5, we can choose a sequence $v_n \subset D(\mathcal{A}_0)$, such that

$$\|u - v_n\|_0 \,+\, \|\mathcal{A}_0 v_n - \mathcal{A}u\|_0 \,\leq\, \frac{1}{n}, \;\; n \geq 1.$$

Now we define $u_n(x) \;=\; O_{\frac{1}{\sqrt{n}}}\, v_n(x)$, $x \in H$, where O_t is the heat semigroup associated with Q.

We know that $u_n \in D(\mathcal{A}_0)$, $n \geq 1$, see Proposition 3.3.4. We show that (i) is satisfyed. We have

$$\begin{aligned} &\|u_n - u\|_0 \leq \|O_{\frac{1}{\sqrt{n}}}\, v_n - O_{\frac{1}{\sqrt{n}}} u\|_0 + \|O_{\frac{1}{\sqrt{n}}} u - u\|_0 \\ &\leq \|v_n - u\|_0 + \|O_{\frac{1}{\sqrt{n}}} u - u\|_0 \end{aligned} \tag{4.3.11}$$

letting $n \to \infty$ in the last term, we find $\lim_{n\to\infty} \|u_n - u\|_0 = 0$, by the strong continuity of O_t. Moreover, by using that

$$\mathcal{A}u_n = O_{\frac{1}{\sqrt{n}}} \mathcal{A}v_n, \quad n \geq 1$$

and repeating the previous estimates, we obtain (i).

As for (ii), we only check that (4.3.10) holds. To this purpose we use the following estimate, see Proposition 3.2.3,

$$\|D_Q^2 O_t f\|_{0,\mathcal{L}_2(H)} \leq \frac{2}{t}\|f\|_0, \quad f \in \mathcal{C}_b(H).$$

There results

$$\begin{aligned}
&\|D_Q^2 u_n - D_Q^2 u\|_{0,\mathcal{L}_2} \\
&\leq \|D_Q^2 O_{\frac{1}{\sqrt{n}}} v_n - D_Q^2 O_{\frac{1}{\sqrt{n}}} u\|_{0,\mathcal{L}_2} + \|D_Q^2 O_{\frac{1}{\sqrt{n}}} u - D_Q^2 u\|_{0,\mathcal{L}_2} \\
&\leq 2\sqrt{n}\,\|v_n - u\|_0 + \|D_Q^2 O_{\frac{1}{\sqrt{n}}} u - D_Q^2 u\|_{0,\mathcal{L}_2} \leq \tfrac{2}{\sqrt{n}} + \|D_Q^2 O_{\frac{1}{\sqrt{n}}} u - D_Q^2 u\|_{0,\mathcal{L}_2}.
\end{aligned} \tag{4.3.12}$$

Now to conclude we remark that, using the Bochner integral as in the proof of Proposition 3.3.1, there results, for any $x \in H$,

$$\begin{aligned}
&\|D_Q^2 O_t u(x) - D_Q^2 u(x)\|_{\mathcal{L}_2(H)} \\
&\leq \int_H \|D_Q^2 u(x + \sqrt{t}y) - D_Q^2 u(x)\|_{\mathcal{L}_2(H)}\ \mathcal{N}(0,Q)dy \\
&\leq \int_H \omega_{D_Q^2 u}(|\sqrt{t}y|)\mathcal{N}(0,Q)dy,
\end{aligned} \tag{4.3.13}$$

where $\omega_{D_Q^2 u}$ denotes the modulus of continuity of $D_Q^2 u$. Letting $t \to 0^+$ in the last term, by the Dominated Convergence Theorem, we find

$$\lim_{t\to 0^+} \|D_Q^2 O_t u - D_Q^2 u\|_{0,\mathcal{L}_2(H)} = 0.$$

Using this fact, letting $n \to \infty$ in the last term of (4.3.12), we find (4.3.10). This completes the proof. ∎

Proof of Theorem 4.3.3 First remark that, possibly replacing u by $u - \inf_{x\in H} u(x)$, we may assume that $u \geq 0$. The proof is divided into two parts.

Step 1. We show that formula (4.3.2) is true if $u \in D(\mathcal{A}_0)$, see (4.3.9).

Let $u \in D(\mathcal{A}_0)$. By Lemma 4.3.4, there exists a map $u_\epsilon \in D(\mathcal{A}_0)$, that attains a maximum on H and such that (4.3.3) holds. There results

$$\lambda u_\epsilon(x) - \mathcal{A}_0 u_\epsilon(x) - \frac{1}{2}\mathrm{Tr}\,[(F(x)D_Q^2 u_\epsilon(x)] = \lambda u_\epsilon(x) - \frac{1}{2}\mathrm{Tr}\,[(1 + F(x))D_Q^2 u_\epsilon(x)] \tag{4.3.14}$$

$$= f(x) + \lambda(u_\epsilon - u)(x) - \frac{1}{2}\text{Tr }[(1+F(x))D_Q^2(u_\epsilon - u)(x)].$$

Let x_ϵ be a point of H, where u_ϵ attains a maximum. Then we have

$$\text{Tr }[D_Q^2 u_\epsilon(x_\epsilon)] = \sum_{k=1}^{\infty} \lambda_h D_{hh}^2 u_\epsilon(x_\epsilon) \leq 0,$$

and moreover

$$\begin{aligned}\text{Tr }[F(x_\epsilon)D_Q^2 u_\epsilon(x_\epsilon)] &= \text{Tr }[F^{1/2}(x_\epsilon)D_Q^2 u_\epsilon(x_\epsilon)F^{1/2}(x_\epsilon)] \\ &= \sum_{k=1}^{\infty} < D_Q^2 u_\epsilon(x_\epsilon)F^{1/2}(x_\epsilon)e_k, F^{1/2}(x_\epsilon)e_k > \leq 0.\end{aligned}$$

Hence we obtain by (4.3.14)

$$\lambda\|u_\epsilon\|_0 = \lambda u_\epsilon(x_\epsilon) \leq \|f\|_0 + M\|u - u_\epsilon\|_{\mathcal{C}_b^2(H)},$$

where $M = M(\lambda, Q, F) > 0$. Letting $\epsilon \to 0^+$ in the last formula, we find (4.3.2), invoking Lemma 4.3.4.

Step 2. General case. Let $u \in D(\mathcal{A}) \cap C_Q^2(H)$. By Lemma 4.3.5, we can choose a sequence $(u_n) \subset D(\mathcal{A}_0)$ such that $u_n \to u$ in $\mathcal{C}_Q^2(H)$, $\mathcal{A}_0 u_n \to \mathcal{A}u$ in $\mathcal{C}_b(H)$ as $n \to \infty$ and further

$$\lim_{n\to\infty} \|D_Q^2 u_n - D_Q^2 u\|_{0,\mathcal{L}_2(H)} = 0$$

We define, for any $n \geq 1$,

$$g_n(x) = \lambda u_n(x) - \frac{1}{2}\text{Tr }[(1+F(x))D_Q^2 u_n(x)], \quad x \in H$$

Now by the estimate

$$|\text{Tr }(F(x)D_Q^2 u_n(x))| \leq \|F(x)\|_{\mathcal{L}_2} \|D_Q^2 u_n(x)\|_{\mathcal{L}_2}, \quad x \in H$$

and since $\mathcal{A}_0 u_n = \frac{1}{2}\text{Tr }(D_Q^2 u_n) \in \mathcal{C}_b(H)$, we easily obtain that $g_n \in \mathcal{C}_b(H)$, $n \geq 1$. Applying the first step we derive that

$$\|u_n\|_0 \leq \frac{1}{\lambda}\|g_n\|_0, \quad n \geq 1. \tag{4.3.15}$$

By our assumptions on u_n, we know that

$$g_n \to (\lambda u - \mathcal{A}u - \frac{1}{2}\text{Tr }[FD_Q^2 u]) = g \text{ in } \mathcal{C}_b(H) \text{ as } n \to \infty.$$

Letting $n \to \infty$ in (4.3.15), we obtain the thesis. The proof is complete. ∎

4.3.2 A priori estimate and continuity method

A priori estimates for (4.3.1) are proved in the next result, that improves Theorem 6.2 of Cannarsa and Da Prato [12].

Theorem 4.3.6 *Assume that F satisfies Hypothesis 4.3.1 and $f \in \mathcal{C}^\theta_Q(H)$. Let u be a solution of (4.3.1). Then there exists a constant $c = c(\lambda, Q, \theta, \|F\|_{\theta,Q}) > 0$, such that:*

$$\|u\|_{2,Q} + \|D^2_Q u\|_{\theta,Q,\mathcal{L}_2(H)} \ \leq\ c\,\|f\|_{\theta,Q}.$$

We need two preliminary Lemmas. The first is a non standard interpolation result of independent interest.

Lemma 4.3.7 *Let $v \in C^2_Q(H)$ such that $D^2_Q v \in \mathcal{C}_b(H, \mathcal{L}_2(H))$. Assume that for any $N \in \mathcal{F}_1$, see (4.2.4), it holds $\mathrm{Tr}[ND^2_Q v] \in D_\mathcal{A}(\theta/2, \infty)$ and*

$$\|D^2_Q v\|_{\theta/2,\mathcal{A},\mathcal{L}_2} \stackrel{\text{def}}{=} \sup_{N\in\mathcal{F}_1} \|\mathrm{Tr}[ND^2_Q v]\|_{\theta/2,\mathcal{A}} < +\infty. \tag{4.3.16}$$

Then, for any $t > 0$, the following interpolatory inequality holds:

$$\|D^2_Q v\|_{0,\mathcal{L}_2} \leq\ C_\theta\,\|v\|_0^{\frac{\theta}{2+\theta}}\ \|D^2_Q v\|^{\frac{2}{2+\theta}}_{\theta/2,\mathcal{A},\mathcal{L}_2} \tag{4.3.17}$$

Proof First notice that, by (4.2.4),

$$\|D^2_Q v\|_{0,\mathcal{L}_2} = \sup_{x\in H}\sup_{N\in\mathcal{F}_1} |\mathrm{Tr}[ND^2_Q v](x)|$$

Note that $O_t(\mathrm{Tr}[ND^2_Q v]) = \mathrm{Tr}[ND^2_Q O_t v]$, see (4.2.14). Then for any $N \in \mathcal{F}_1$ and $t > 0$ we have:

$$\begin{aligned}
\|\mathrm{Tr}[ND^2_Q v]\|_0 \ &\leq\ \left\|O_t\left(\mathrm{Tr}[ND^2_Q v]\right) - \mathrm{Tr}[ND^2_Q v]\right\|_0 + \|O_t(\mathrm{Tr}[ND^2_Q v])\|_0 \\
&\leq\ t^{\frac{\theta}{2}}\|\mathrm{Tr}[ND^2_Q v]\|_{\theta/2,\mathcal{A}} + \|\mathrm{Tr}[ND^2_Q O_t v]\|_0 \\
&\leq\ t^{\frac{\theta}{2}}\sup_{N\in\mathcal{F}_1}\|\mathrm{Tr}[ND^2_Q v]\|_{\theta/2,\mathcal{A}} + \|D^2_Q O_t v\|_{0,\mathcal{L}_2} \\
&\leq\ t^{\frac{\theta}{2}}\|D^2_Q v\|_{\theta/2,\mathcal{A},\mathcal{L}_2} + \frac{2}{t}\|v\|_0
\end{aligned}$$

In the last passage we have used Proposition 3.2.3. Taking the infimum over $t > 0$ in the last term, we obtain the thesis. ∎

Let $\eta \in C^\infty_b([0, +\infty[)$ be such that

$$0 \leq \eta \leq 1\ ;\ \eta(r) = 1,\ r \leq 1\ ;\quad \eta(r) = 0,\ r \geq 2\ ;$$

and define for any $x \in H$, $r > 0$,

$$\rho_{x,r}(z)\ =\ \eta(\frac{|z-x|^2}{r^2}),\quad z \in H. \tag{4.3.18}$$

Clearly $\rho_{x,r} \in C_b^\infty(H)$ ([1]) and moreover

$$0 \le \rho_{x,r} \le 1 \qquad \rho_{x,r}(z) = \begin{cases} 1 & \text{if } z \in B(x,r) \\ 0 & \text{if } z \notin B(x,2r) \end{cases}$$

It is easy to prove the next result.

Lemma 4.3.8 *Let $u \in D(\mathcal{A})$. Then $\rho_{x,r}u \in D(\mathcal{A})$, for any $x \in H$, $r > 0$. Moreover*

$$\mathcal{A}(\rho_{x,r}u) = \rho_{x,r}\mathcal{A}u + \langle D_Q u, Q^{1/2}D\rho_{x,r}\rangle + \frac{1}{2}\mathrm{Tr}[QD^2\rho_{x,r}]. \tag{4.3.19}$$

Proof Assume first that $u \in D(\mathcal{A}_0)$, see (4.3.9). Then it is easy to check that $\rho_{x,r}u \in D(\mathcal{A}_0)$ and moreover, denoting by $\hat{D}^2u$ the second Hadamard derivative of u,

$$\begin{aligned}
\mathcal{A}_0(\rho_{x,r}u)(z) &= \frac{1}{2}\mathrm{Tr}\ (Q^{1/2}\hat{D}^2(\rho_{x,r}u)(z)Q^{1/2}) \\
&= \frac{1}{2}\sum_{k\ge 1} < \hat{D}^2(\rho_{x,r}u)(z)Q^{1/2}e_k, Q^{1/2}e_k >= \\
\frac{1}{2}u(z)\mathrm{Tr}\ (Q\hat{D}^2\rho_{x,r}(z)) &+ \frac{1}{2}\rho_{x,r}(z)\mathrm{Tr}\ (Q\hat{D}^2u(z)) \\
&\quad + < D_Qu(z), D_Q\rho_{x,r}(z) > \\
= \rho_{x,r}\mathcal{A}_0(u)(z) + u\mathcal{A}_0\rho_{x,r}(z) &+ < D_Qu(z), D_Q\rho_{x,r}(z) >, \quad z \in H.
\end{aligned}$$

Now if $u \in D(\mathcal{A})$ there exists a sequence $\{u_n\} \subset D(\mathcal{A}_0)$, see Theorem 3.3.5, such that

$$u_n \to u \ \text{ and } \ \mathcal{A}_0u_n \to \mathcal{A}u \ \text{ in } C_b(H).$$

It follows that, for any $n \ge 1$,

$$\mathcal{A}_0(\rho_{x,r}u_n) = \rho_{x,r}\mathcal{A}_0u_n + u_n\mathcal{A}_0\rho_{x,r} + < D_Qu_n, D_Q\rho_{x,r} > .$$

Now letting $n \to \infty$, we obtain the assertion. Note that, by formula 3.4.3,

$$\|D_Q(u_n - u)\|_0 \le C\,\|u - u_n\|_0^{1/2}\,\|\mathcal{A}(u_n - u)\|_0^{1/2}, \quad n \ge 1. \ \blacksquare$$

Proof of Theorem 4.3.6 Let $f \in C_Q^\theta(H)$ and let u be a solution of equation (4.3.1). We consider the maps $\rho_{x,r}$ defined in (4.3.18). Note that, setting $\rho_{x,r} = \rho$, we readily find

$$D\rho(z) = 2\eta'\left(\frac{|z-x|^2}{r^2}\right)\frac{z-x}{r^2} \ \text{ and}$$

$$D^2\rho(z) = 4\eta''\left(\frac{|z-x|^2}{r^2}\right)\frac{z-x}{r^2}\otimes\frac{z-x}{r^2} + \eta'\left(\frac{|x-z|^2}{r^2}\right)I_H, \quad z \in H,$$

[1] $\mathcal{C}_b^\infty(H)$ denotes the subspace of $\mathcal{C}_b(H)$ of all functions having uniformly continuous and bounded Fréchet derivatives of any order.

where for any $a, b \in H$, $a \otimes b(u,v) \stackrel{\rm def}{=} < a, u >< b, v >$, $u, v \in H$ and I_H is the identity. By the previous formula it follows that $\|D\rho\|_0 \le \frac{4}{r}\,\|\eta'\|_0$ and

$$\|D^2\rho\|_{0,\mathcal{L}(H)} \le \frac{16}{r^2}(\|\eta''\|_0 + \|\eta'\|_0).$$

Fix $x_0 \in H$, $r > 0$ and set $v = \rho_{x_0,r}u = \rho u$. We shall denote by C_i, $i \in \mathbb{N}$, constants depending only on λ, Q, θ, F. By Lemma 4.3.8 we have

$$\lambda v - \mathcal{A}v - \frac{1}{2}\mathrm{Tr}[F(x_0)D_Q^2 v] = f_1 + f_2 + f_3,$$

where

$$f_1(x) = \rho(x)f(x), \qquad f_2(x) = \frac{1}{2}\mathrm{Tr}[(F(x) - F(x_0))D_Q^2 v(x)]$$

$$f_3(x) = -\langle (I + F(x))D_Q u(x), D_Q\rho(x)\rangle - u(x)\frac{1}{2}\mathrm{Tr}[(I + F(x))D_Q^2\rho(x)]$$

By Theorem 4.2.5 we have, by using (4.3.16),

$$\|v\|_{2,Q} + \|D_Q^2 v\|_{\theta/2,\mathcal{A},\mathcal{L}_2} \le C(\|f_1\|_{\theta,Q} + \|f_2\|_{\theta,Q} + \|f_3\|_{\theta,Q}). \tag{4.3.20}$$

We will frequently use the simple estimate

$$\|lg\|_{\theta,Q} \le \|l\|_0[g]_{\theta,Q} + \|g\|_0[l]_{\theta,Q}, \quad l,\, g \in \mathcal{C}_Q^\theta(H).$$

First notice that

$$\|f_1\|_{\theta,Q} \le K_r\,\|f\|_{\theta,Q}. \tag{4.3.21}$$

Let us estimate $\|f_2\|_{\theta,Q}$. First we have

$$\|f_2\|_0 \le C_1\|D_Q^2 v\|_{0,\mathcal{L}_2(H)}.$$

Then, denoting by ω_F the modulus of continuity of F, there results

$$\begin{aligned}
[f_2]_{\theta,Q} &\le C_2\left(\sup_{x\in B(x_0,2r)}\|F(x) - F(x_0)\|_{\mathcal{L}_2}[D_Q^2 v]_{\theta,Q,\mathcal{L}_2} + M\|D_Q^2 v\|_{0,\mathcal{L}_2}\right)\\
&\le C_3\left(\omega_F(2r)\left[D_Q^2 v\right]_{\theta,Q,\mathcal{L}_2} + M\|D_Q^2 v\|_{0,\mathcal{L}_2}\right).
\end{aligned}$$

By Lemma 4.3.7 and by (4.2.16) it follows that

$$\begin{aligned}
\|f_2\|_{\theta,Q} &\le C_4\left(\omega_F(2r)\left[D_Q^2 v\right]_{\theta,Q,\mathcal{L}_2} + \|D_Q^2 v\|_{0,\mathcal{L}_2}\right)\\
&\le C_{41}\left(\omega_F(2r)\left[D_Q^2 v\right]_{\theta,Q,\mathcal{L}_2} + \|v\|_0^{\frac{\theta}{2+\theta}}\|D_Q^2 v\|_{\theta/2,\mathcal{A},\mathcal{L}_2}^{\frac{2}{2+\theta}}\right)\\
&\le C_5\left((\omega_F(2r) + r^{\theta/2})\|D_Q^2 v\|_{\theta/2,\mathcal{A},\mathcal{L}_2} + \frac{1}{r}\|v\|_0\right).
\end{aligned}$$

Using the maximum principle, see Theorem 4.3.3, we obtain

$$\|f_2\|_{\theta,Q} \le C_6 \left((\omega_F(2r) + r^{\theta/2}) \|D_Q^2 v\|_{\theta/2,\mathcal{A},\mathcal{L}_2} + \frac{1}{r}\|f\|_0 \right). \tag{4.3.22}$$

As for $\|f_3\|_{\theta,Q}$, we easily obtain the following estimate:

$$\|f_3\|_{\theta,Q} \le C_7(\|f\|_{\theta,Q} + E_r\|u\|_{1+\theta,Q}). \tag{4.3.23}$$

Collecting (4.3.20)—(4.3.23) we deduce

$$\begin{aligned} &\|v\|_{2,Q} + \|D_Q^2 v\|_{\theta/2,\mathcal{A},\mathcal{L}_2} \le C\left(\|f_1\|_{\theta,Q} + \|f_2\|_{\theta,Q} + \|f_3\|_{\theta,Q}\right) \\ &\le C_8 \left((\omega_F(2r) + r^{\theta/2}) \|D_Q^2 v\|_{\theta/2,\mathcal{A},\mathcal{L}_2} + \left(1 + \frac{1}{r}\right) \|f\|_{\theta,Q} + E_r\|u\|_{1+\theta,Q} \right). \end{aligned}$$

Now we choose $r > 0$ such that $C_8(\omega_F(2r) + r^{\theta/2}) < 1/2$. This way, by using also (4.2.16), we infer

$$\begin{aligned} &\|v\|_{2,Q} + \|D_Q^2 v\|_{\theta,Q,\mathcal{L}_2} \le \|v\|_{2,Q} + 2\|D_Q^2 v\|_{\theta/2,\mathcal{A},\mathcal{L}_2} \\ &\le C_9\left(\|f\|_{\theta,Q} + \|u\|_{1+\theta,Q}\right). \end{aligned}$$

Since $v = \rho u$, we obtain

$$\begin{aligned} &\|u\|_{C_Q^2(B(x_0,r))} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}_2\,(B(x_0,r))} \le \|v\|_{2,Q} + \|D_Q^2 v\|_{\theta,Q,\mathcal{L}_2(H)} \\ &\le C_9\left(\|f\|_{\theta,Q} + \|u\|_{1+\theta,Q}\right). \end{aligned}$$

Since C_9 is independent of x_0, it follows that

$$\|u\|_{2,Q} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}_2} \le C_9(\|f\|_{\theta,Q} + \|u\|_{1+\theta,Q}).$$

Now notice that, in a standard way, one proves that

$$\begin{aligned} &\|u\|_{1+\theta,Q} \le C_{11}\|u\|_0^{\frac{1}{2+\theta}} \left(\|u\|_{2,Q} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}(H)}\right)^{\frac{1+\theta}{2+\theta}} \\ &\le C_{11}\,\|u\|_0^{\frac{1}{2+\theta}} \left(\|u\|_{2,Q} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}_2(H)}\right)^{\frac{1+\theta}{2+\theta}}, \end{aligned}$$

from which, using the Young inequality ([2]), there results

$$\|u\|_{2,Q} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}_2} \le C_{12}(\|f\|_{\theta,Q} + K_\varepsilon\|u\|_0 + \varepsilon(\|u\|_{2,Q} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}_2(H)}).$$

Choosing ε small enough and using again the maximum principle, we finally get

$$\|u\|_{2,Q} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}_2} \le C_{13}\,\|f\|_{\theta,Q}$$

as required. The proof is complete. ∎

From Theorem 4.3.6 we can deduce the next final result, that :

Theorem 4.3.9 *Assume that F fulfills Hypothesis 4.3.1 and let $f \in C_Q^\theta(H)$. Then there exists a unique solution (see Definition 4.3.2) of equation (4.3.1).*

[2]If $a, b \ge 0$ and $p, q > 1$ such that $1/p + 1/q = 1$, we have $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.

Proof Uniqueness is an immediate consequence of the maximum principle, see Theorem 4.3.3. In order to prove existence, we can adapt, without difficulties, the classical continuity method.

To this aim let us introduce the set Λ consisting of all $\alpha \in [0,1]$ such that equation

$$\lambda u(x) - \mathcal{A}u(x) - \frac{\alpha}{2}\mathrm{Tr}\,[F(x)D_Q^2u(x)] = f(x), \quad x \in H, \tag{4.3.24}$$

has a unique solution for all $f \in C_Q^\theta(H)$. In view of Theorem 4.2.2 we have that $0 \in \Lambda$. We will show that $\Lambda = [0,1]$. This fact implies that equation (4.3.1), corresponding to $\alpha = 1$, has a unique solution.

We first prove that the set Λ is *closed* in $[0,1]$.

Let $\{\alpha_n\} \subset \Lambda$ be a sequence convergent to some element α_0, and let $\{u_n\}$ be the solutions of equations

$$\lambda u_n(x) - \mathcal{A}u_n(x) - \frac{\alpha_n}{2}\,\mathrm{Tr}\,[F(x)D_Q^2u_n(x)] = f(x), \;\; x \in H. \tag{4.3.25}$$

First we remark that it holds, for any $n \geq 1$,

$$\begin{aligned}&\|\mathrm{Tr}\,(FD_Q^2u_n)\|_{\theta,Q}\\ &\leq\; [F]_{\theta,Q,\mathcal{L}_2}\,\|D_Q^2u_n\|_{0,\mathcal{L}_2} \,+\, [D_Q^2u_n]_{\theta,Q,\mathcal{L}_2}\|F\|_{0,\mathcal{L}_2} \,\leq 2\|F\|_{\theta,Q,\mathcal{L}_2}\,\|D_Q^2u_n\|_{\theta,Q,\mathcal{L}_2}.\end{aligned} \tag{4.3.26}$$

We introduce the Banach space $Y = \{g \in \mathcal{C}_Q^2(H)$, such that $D_Q^2g \in \mathcal{C}_Q^\theta(H,\mathcal{L}_2(H))\}$, endowed with the norm

$$\|g\|_Y = \|g\|_{2,Q} \,+\, \|D_Q^2g\|_{\theta,Q,\mathcal{L}_2(H)}, \;\; g \in Y.$$

We claim that $\{u_n\}$ is a Cauchy sequence in Y. Indeed from the identity

$$\lambda(u_n-u_m) - \mathcal{A}(u_n-u_m) - \frac{\alpha_n}{2}\,\mathrm{Tr}\,[F(x)D_Q^2(u_n-u_m)] = \frac{(\alpha_n-\alpha_m)}{2}\,\mathrm{Tr}\,[F(x)D_Q^2u_m],$$

applying Theorem 4.3.6 and (4.3.26), it follows that there exists $C_1 > 0$ such that

$$\begin{aligned}&\|u_n-u_m\|_{2,Q} \,+\, \|D_Q^2(u_n-u_m)\|_{\theta,Q,\mathcal{L}_2} \leq C\,\frac{|\alpha_n-\alpha_m|}{2}\,\|\;\mathrm{Tr}\,[F(\cdot)D_Q^2u_m(\cdot)\|_{\theta,Q}\\ &\leq\; 2C\,\frac{|\alpha_n-\alpha_m|}{2}\,\|F\|_{\theta,Q,\mathcal{L}_2}\,\|D_Q^2u_m(\cdot)\|_{\theta,Q,\mathcal{L}_2} \leq C_1|\alpha_n-\alpha_m|\,\|F\|_{\theta,Q,\mathcal{L}_2}\,\|f\|_{\theta,Q}.\end{aligned}$$

Denote by $u_0 \in Y$, the limit of (u_n) in Y. Using that $\mathcal{A}$ is a closed operator and passing to the limit, as $n \to \infty$, in (4.3.25) it is easy to check that the u_0 is the solution of

$$\lambda u_0(x) - \mathcal{A}u_0(x) - \frac{\alpha_0}{2}\,\mathrm{Tr}\,[F(x)D_Q^2u_0(x)] = f(x).$$

Hence $u_0 \in \Lambda$ and Λ is closed.
It remains to prove that Λ is *open* in $[0, 1]$.

Let $\alpha_0 \in \Lambda$, and let u_0 be the corresponding solution to (4.3.24). We are going to show that equation (4.3.24) has a solution for any α close enough to α_0. We write (4.3.24) as

$$\begin{aligned}&\lambda u(x) - \mathcal{A}u(x) - \tfrac{\alpha_0}{2}\mathrm{Tr}\,[F(x)D_Q^2 u(x)]\\&= \tfrac{\alpha-\alpha_0}{2}\,\mathrm{Tr}\,(F(x)D_Q^2 u(x)) + f(x), \quad x \in H.\end{aligned} \tag{4.3.27}$$

Now we denote by $T_{\alpha_0}v, \ v \in \mathcal{C}_Q^\theta(H)$, the unique solution ϕ of

$$\lambda\phi(x) - \mathcal{A}\phi(x) - \frac{\alpha_0}{2}\mathrm{Tr}\,[F(x)D_Q^2\phi(x)] = v(x), \quad x \in H.$$

Moreover we introduce the linear map $\gamma : \mathcal{C}_Q^\theta(H) \to \mathcal{C}_Q^\theta(H)$,

$$\gamma(v)(x) \ = \ \frac{\alpha-\alpha_0}{2}\,\mathrm{Tr}\,(F(x)D_Q^2 T_{\alpha_0}v(x)), \ \ v \in \mathcal{C}_Q^\theta(H), \ x \in H.$$

We can define such a map γ, since for any $v \in \mathcal{C}_Q^\theta(H)$, $D_Q^2 T_{\alpha_0}v \in \mathcal{C}_Q^\theta(H, \mathcal{L}_2(H))$ and invoking Theorem 4.3.6, we easily obtain

$$\begin{aligned}&\|\mathrm{Tr}\,(F(\cdot)D_Q^2 T_{\alpha_0}v(\cdot))\|_{\theta,Q}\\&\leq [F]_{\theta,Q,\mathcal{L}_2}\,\|D_Q^2 T_{\alpha_0}v\|_{0,\mathcal{L}_2} + [D_Q^2 T_{\alpha_0}v]_{\theta,Q,\mathcal{L}_2}\|F\|_{0,\mathcal{L}_2}\\&\leq C\|F\|_{\theta,Q,\mathcal{L}_2}\,\|v\|_{\theta,Q}.\end{aligned} \tag{4.3.28}$$

This way (4.3.27) becomes: $\ v - \gamma(v) = f$. Note that, by (4.3.28),

$$\|\gamma(v)\|_{\theta,Q} \ \leq \ C\frac{\alpha-\alpha_0}{2}\|F\|_{\theta,Q,\mathcal{L}_2}\,\|v\|_{\theta,Q},$$

where $v \in \mathcal{C}_Q^\theta(H)$. Using the contraction principle, we derive that (4.3.24) has a unique solution provided that $|\alpha-\alpha_0|$ is small enough. The proof is complete. ∎

Chapter 5

Schauder estimates for a homogeneous infinite dimensional Dirichlet problem in a half space of a Hilbert space.

5.1 Introduction

In this chapter we are concerned with the following infinite dimensional Dirichlet problem:

$$\begin{cases} \lambda\psi(x) \; - \; \dfrac{1}{2}\mathrm{Tr}\,[QD^2\psi(x)] = f(x), \quad x \in H_+, \;\; \lambda > 0, \\ \psi(z) = 0, \qquad z \in \partial H_+, \end{cases} \tag{5.1.1}$$

where $H_+ \;=\; \{x \in H, \;\; < x, e_1 > \, > 0\,\}$, H is a real separable Hilbert space (inner product $< \cdot, \cdot >$ and norm $| \cdot |$), $e_1 \in H$, f, ψ are real functions defined on $\overline{H_+}$, $f \in \mathcal{C}_b(H_+)$ and Q is a positive self-adjoint trace class operator in H (Tr (Q) denotes the trace of Q), having e_1 as eigenvector.

Here we study existence and uniqueness for a solution ψ of the problem (5.1.1). Moreover we give an optimal regularity result of Schauder-type for ψ. We follow Priola [66], with in addition some improvements.

Schauder estimates for equation (5.1.1) on the whole space H, also with variable coefficients, have been first obtained in Cannarsa and Da Prato [12] and then improved in Priola and Zambotti [70], see Chapter 4. Here we extend these results to the half space H_+.

We first construct, using the Gaussian measure, a semigroup P_t on $\mathcal{C}_b(H_+)$ naturally related to the Dirichlet problem (5.1.1). It turns out that P_t is not a strongly continuous semigroup on $\mathcal{C}_b(H_+)$, contrary to the heat semigroup on $\mathcal{C}_b(H)$ used to study (4.1.1) in Chapter 4. However a generator $\mathcal{T}$ for P_t can be defined, see Proposition 5.2.8, by Laplace transform

$$(\lambda - \mathcal{T})^{-1} f\,(x) \stackrel{\text{def}}{=} \int_0^\infty e^{-\lambda t}\; P_t f(x)\; dt, \quad f \in \mathcal{C}_b(H_+), \;\; x \in H_+, \; \lambda > 0, \tag{5.1.2}$$

as in Cerrai [14] and we are able to find a *core* for $\mathcal{T}$, see Theorem 5.2.13. When a function ϕ belongs to this core, we prove that $\mathcal{T}\phi = \frac{1}{2}\mathrm{Tr}\,[QD^2\phi]$. It is also possible to define a notion of strict solution for (5.1.1) (see Definition 5.2.9) and we can show that if the datum f in (5.1.1) is smooth, then there exists a unique strict solution, see Theorem 5.2.11. Otherwise for any datum $f \in \mathcal{C}_b(H_+)$, there exists always a strong solution (see Corollary 5.2.14).

We study optimal regularity of the strong solution. To this purpose we need to consider differentiability of functions along the subspace $Q^{1/2}H$ (see Section 1.3 for a precise definition). We denote by Q-differentiability this type of differentiation and consider the associated spaces of functions $\mathcal{C}^1_Q(H_+)$, $\mathcal{C}^2_Q(H_+)$, subspaces of $\mathcal{C}_b(H_+)$. Moreover D^1_Q (resp. D^2_Q) denotes the first (resp. second) derivative with respect to $Q^{1/2}H$. Obviously $Q-$differentiability is less restrictive than usual Fréchet differentiability.

We also consider Hölder continuous functions with respect to $Q^{1/2}H$ and call them Q-Hölder continuous functions. Our main result about problem (5.1.1), see Theorem 5.3.16, states that when the datum f is Q-Hölder continuous on H_+ and Hölder continuous (in the usual meaning) on ∂H_+ then, denoting by ψ the strong solution depending of f, we have that $\psi \in \mathcal{C}^2_Q(H_+)$ and the second $Q-$derivative $D^2_Q\psi : \overline{H_+} \to \mathcal{L}_2(H)$ is Q-Hölder continuous on H_+ and Hölder continuous on ∂H_+; further Schauder estimates hold.

However when the datum f is only Q-Hölder continuous we cannot prove that the strong solution ψ is such that $D^2_Q\psi$ is Q-Hölder continuous (compare with Cannarsa and Da Prato [12] and Theorem 4.2.2), we give only partial results. Finally the results of this chapter improve those in Priola [66]. Indeed here, taking into account the results in Priola and Zambotti [70], we are able to prove that the second Q-derivative of ψ at any $x \in H_+$ is of Hilbert-Schmidt type (on this subject we refer to Introduction in Chapter 4).

We recall some notations, see Chapter 1 for more details. $\mathcal{L}_1(H)$ denotes the subspace of $\mathcal{L}(H)$ ([1]) of all trace class operators. Moreover $\mathcal{L}_2(H)$ denotes the subspace of $\mathcal{L}(H)$ of all Hilbert-Schmidt operators.

Let Q be a positive self-adjoint trace class operator in H, we fix once and for all an orthonormal basis of H, $\{e_k\}_{k\geq 1}$, that diagonalizes Q: for any $x \in H$,
$Qx = \sum_{k=1}^{\infty} \lambda_k x_k e_k$ with $x_k =< x, e_k >$.

In the following we indentify each element x of H with its coordinates with respect to the basis $\{e_k\}_{k\geq 1}$.

Let H' be the Hilbert subspace of H generated by $\{e_k\}_{k\geq 2}$. We set
$Q'x' = \sum_{k=2}^{\infty} \lambda_k x'_k e_k$, $x' = (x'_k) \in H'$.

We define the open half space H_+ :

$$H_+ \stackrel{\text{def}}{=} \{\ x = (x_1, x') \in H \ \text{ such that } \ x_1 > 0, \ \ x' \in H'\ \},$$

[1] $\mathcal{L}(H)$ stands for the Banach space of all bounded linear operators on H, endowed with the usual operator norm.

that we identify with $\mathbb{R}_+ \times H'$ ($\mathbb{R}_+ = (0,\infty)$). Furthermore we define:

$$\begin{aligned} H_- &= \{\, x = (x_1, x') \in H \ \text{ such that } \ x_1 < 0, \ \ x' \in H' \,\} \\ \partial H_+ &= \{\, x = (0, x') \in H \ \text{ such that } \ x' \in H' \,\} \\ \overline{H}_+ &= \ H_+ \bigcup \partial H_+. \end{aligned}$$

We recall some basic functions spaces that will be used.

Let Ω be an open set of H and $(E, \|\cdot\|_E)$ be a real Banach space, $\mathcal{C}_b(\Omega, E)$ stands for the Banach space of all uniformly continuous and bounded functions $f : \Omega \to E$, endowed with the sup norm $\|\cdot\|_0$ (i.e. $\|f\|_0 = \sup_{x\in\Omega} \|f(x)\|_E$.).

$\mathcal{C}^\theta_b(\Omega, E)$, $\theta \in (0,1)$, denotes the space of all functions in $\mathcal{C}_b(\Omega, E)$, which are θ-Hölder continuous.

Let $f \in \mathcal{C}_b(\Omega, E)$, the modulus of continuity of f will be indicated by ω_f. Moreover the uniform continuity of f allows us to consider values of f on $\partial\Omega$ and implies that $\mathcal{C}_b(\Omega, E) = \mathcal{C}_b(\overline{\Omega}, E)$.

When $E = \mathbb{R}$, we set $\mathcal{C}_b(\Omega) = \mathcal{C}_b(\Omega, \mathbb{R})$ and $\mathcal{C}^\theta_b(\Omega) = \mathcal{C}^\theta_b(\Omega, \mathbb{R})$, $\theta \in (0,1)$.

We identify $\mathcal{C}_b(H')$ with a subspace of $\mathcal{C}_b(H_+)$ through the following embedding

$$J : \mathcal{C}_b(H') \to \mathcal{C}_b(H_+), \ \ Jh\,(x_1, x') \stackrel{\text{def}}{=} h(x'), \quad h \in \mathcal{C}_b(H'), \ x = (x_1, x') \in H_+. \quad (5.1.3)$$

In the sequel for any $h \in \mathcal{C}_b(H')$, we simply write h instead of Jh.

We recall that $\mathcal{N}(x_1, t\lambda_1)$, $x_1 \in \mathbb{R}$, $t > 0$, $\lambda_1 > 0$ stands for the Gaussian measure in $\mathbb{R}$ with density

$$\frac{1}{\sqrt{2\pi t\lambda_1}}\, e^{-\frac{(x_1-y_1)^2}{2t\lambda_1}},$$

with respect to the Lebesgue measure dy_1. Similarly $\mathcal{N}(x, tM)$ is the Gaussian measure in H with mean $x \in H$ and covariance operator tM, that is a non negative self-adjoint trace class operator on H.

We denote by O_t the heat semigroup on $\mathcal{B}_b(H)$ ([2]), associated with the operator Q of the problem (5.1.1),

$$O_t f(x) = \int_H f(x+y)\, \mathcal{N}(0, tQ) dy, \ \ f \in \mathcal{B}_b(H), \ x \in H, \ t > 0, \qquad (5.1.4)$$

see also (3.2.3). It is well known that $O_t\,(\mathcal{C}_b(H)) \subset \mathcal{C}_b(H)$, $t \geq 0$. Moreover the restriction of O_t to $\mathcal{C}_b(H)$, that we still denote by O_t, is a $\mathcal{C}_0$-semigroup.

5.2 Solution of the Dirichlet problem

5.2.1 Construction of P_t

We define the reflection with respect to x_1.

[2]Let A be a Borel set of H, $\mathcal{B}_b(A)$ denotes the Banach space of all real, bounded and Borel functions on A, endowed with the sup norm.

$$\phi_1 : H \to H, \quad \phi_1[(x_1, x')] \stackrel{\text{def}}{=} (-x_1, x'), \quad (x_1, x') \in H. \tag{5.2.1}$$

It is simple to verify that it holds:

$$\phi_1^* = \phi_1, \quad (\phi_1)^2 = Id_H, \quad Q\phi_1 = \phi_1 Q.$$

We shall use the operator $E : \mathcal{B}_b(\overline{H}_+) \to \mathcal{B}_b(H)$,

$$Ef(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & if \;\; x \in \overline{H}_+ \\ \\ -f(\phi_1(x)) & if \;\; x \in H_-. \end{cases} \tag{5.2.2}$$

For any $f \in \mathcal{B}_b(\overline{H}_+)$. Clearly $Ef \in \mathcal{B}_b(H)$ and E is an isometry.
We denote by Rf the restriction of $f \in \mathcal{B}_b(H)$ to $\overline{H}_+$.

Some basic connections among R, E, ϕ_1 and the heat semigroup O_t will be stated in the next lemma.

Lemma 5.2.1 *The following statements hold for* $t \geq 0$*:*
(i) $O_t[g \circ \phi_1](x) = O_t g(\phi_1(x)), \quad x \in H, \quad g \in \mathcal{B}_b(H)$*;*
(ii) $[Ef \circ \phi_1](y) = -Ef(y), \quad f \in \mathcal{B}_b(\overline{H}_+), \quad y \notin \partial H_+$*;*
(iii) $E(RO_tEf)(x) = O_tEf(x), \quad f \in \mathcal{B}_b(\overline{H}_+), \quad x \in H.$

Proof (i) By changing variable, see (1.1.13), we obtain for any $g \in \mathcal{B}_b(H)$, $x \in H$,

$$O_t[g \circ \phi_1](x) = \int_H g(\phi_1(y))\, \mathcal{N}(x, tQ) dy$$
$$= \int_H g(y)\, \mathcal{N}(\phi_1(x), tQ) dy = O_t g(\phi_1(x)).$$

(ii) Let $f \in \mathcal{B}_b(\overline{H}_+)$. If $y \in H_+$, we have $Ef \circ \phi_1(y) = Ef(\phi_1(y)) = -f(y) = -Ef(y)$. If $y \in H_-$ it holds

$$Ef \circ \phi_1(y) = Ef(\phi_1(y)) = f(\phi_1(y)) = -Ef(y).$$

(iii) Let $f \in \mathcal{B}_b(\overline{H}_+)$. Applying (i) and (ii), we find readily

$$E(RO_tEf)(x) = RO_tEf(x) = O_tEf(x), \quad \text{for} \;\; x \in \overline{H}_+.$$

$$E(RO_tEf)(x) = -RO_tEf(\phi_1(x)) = -O_tEf(\phi_1(x)) =$$
$$= -O_t(Ef \circ \phi_1)(x) = O_tEf(x), \;\; x \in H_-. \;\blacksquare$$

We define the following family of operators:

$$\begin{cases} P_t : \mathcal{B}_b(\overline{H}_+) \to \mathcal{B}_b(\overline{H}_+), \quad t \geq 0, \\ P_t \stackrel{\text{def}}{=} RO_tE \end{cases} \tag{5.2.3}$$

The restriction of P_t to $\mathcal{C}_b(H_+)$ turns out to be a semigroup of contractions on $\mathcal{C}_b(H_+)$ as the following proposition shows.

Proposition 5.2.2 *It holds:*
(i) $P_t, \quad t \geq 0,$ *is a semigroup of contractions on* $\mathcal{B}_b(\overline{H}_+)$;
(ii) $P_t(\mathcal{C}_b(H_+)) \subset \mathcal{C}_b(H_+), \quad t \geq 0.$
(iii) *for any* $f \in \mathcal{C}_b(H_+)$ *there exists a continuous function* $\theta_f : [0,\infty) \to [0,\infty)$ *with* $\theta_f(0) = 0$ *such that:* $\omega_{P_t f}(r) \leq \theta_f(r)\,(1 + \frac{1}{\sqrt{t}})$, $r \geq 0, \ t > 0.$

Proof *(i)* We first prove the semigroup property for P_t. Using (ii) of Lemma 5.2.1, we get for any $f \in \mathcal{B}_b(\overline{H}_+)$,

$$P_t P_s f \ = \ (RO_t E)\,(RO_s E) f \ = \ RO_t\,(ERO_s Ef) \ = \ RO_t O_s Ef \ = \ RO_{t+s} Ef \ = \ P_{t+s} f.$$

Moreover P_t is a family of contractions, indeed it holds, for any $f \in \mathcal{C}_b(H_+)$,

$$\|P_t f\|_0 \ \leq \|O_t E f\|_0 \ \leq \|Ef\|_0 \ = \|f\|_{0,}, \quad f \in \mathcal{C}_b(H), \ t \geq 0.$$

(ii) Let $f \in \mathcal{C}_b(H_+)$. We fix $x = (x_1, x')$ and $z = (z_1, z')$ in H_+ such that $x_1 \leq z_1$. Then we have we have, for any $t > 0$,

$$|P_t f(x) - P_t f(z)| \ \leq \int_H |Ef(x+y) - Ef(z+y)|\, \mathcal{N}(0, tQ) dy$$

$$\leq \int_{[-x_1, +\infty)\times H'} |f(x+y) - f(z+y)|\, \mathcal{N}(0,tQ)dy + \int_{(-\infty, -z_1)\times H'} |f(\phi_1(x+y)) - f(\phi_1(z+y))|\, \mathcal{N}(0,tQ)dy + \int_{[-z_1,-x_1))\times H'} |Ef(x+y) - Ef(z+y)|\, \mathcal{N}(0,tQ)dy$$

$$\leq \ 2\omega_f(|x-z|) + \ 2\|f\|_0 \mathcal{N}(0, t\lambda_1)([-z_1, -x_1]) \ \leq 2\omega_f(|x-z|) + 2\frac{\|f\|_0}{\sqrt{2\pi\lambda_1}} \frac{|x_1 - z_1|}{\sqrt{t}}.$$

Now (ii) follows easily. In order to obtain (iii) we set $\theta_f(r) \stackrel{\text{def}}{=} 2\omega_f(r) + 2\frac{\|f\|_0}{\sqrt{2\pi\lambda_1}}\, r, \ \ r \geq 0.$ ∎

We still denote the restriction of P_t to $\mathcal{C}_b(H_+)$ with P_t.
The next proposition expresses a "weak" continuity for the semigroup P_t. In a successive proposition we will prove that P_t is not a strongly continuous semigroup on $\mathcal{C}_b(H_+)$.

Proposition 5.2.3 *Let* $f \in \mathcal{C}_b(H_+)$, *for any* $x \in H_+$, *the mapping:* $[0,\infty) \to \mathbb{R}$, $t \to P_t f(x)$ *is continuous.*

Proof We consider that

$$P_t f(x) \ = \ \int_H Ef(x + \sqrt{t}\,y)\, \mathcal{N}(0,Q)dy, \ \ f \in \mathcal{C}_b(H_+), \ t > 0, \ x \in H_+$$

and the assertion follows by the Dominated Convergence Theorem, taking into account that $\mathcal{N}(0,Q)\,(\partial H_+) = 0$. ■

Now we obtain an integral representation for P_t (see Section 1.1 for notations).

Proposition 5.2.4 *For every* $f \in \mathcal{C}_b(H_+)$, $t>0$, $x \in H_+$ *it holds:*

$$P_t f(x) = \int_{H_+} f(y_1, y')\, D(x_1, t\lambda_1) \otimes \mathcal{N}(x', tQ')(dy_1, dy') \tag{5.2.4}$$

where H_+ *is identified with* $\mathbb{R}_+ \times H'$ *and* $D(x_1, t\lambda_1) \stackrel{\text{def}}{=} \mathcal{N}(x_1, t\lambda_1) - \mathcal{N}(-x_1, t\lambda_1)$ *is a finite (positive) measure on* $\mathbb{R}_+$ *with mass* $m_{(x_1)} < 1$, *for any* $x_1 > 0$.

Proof Consider the measure $\mathcal{N}(0, tQ)$ restricted to $\mathcal{B}(H_+)$ ([3]). We have that for any $z \in H$, $t>0$, $\mathcal{B}(H_+) = \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(H')$ and $\mathcal{N}(z, tQ) = \mathcal{N}(z_1, t\lambda_1) \otimes \mathcal{N}(z', tQ')$ on $\mathcal{B}(H_+)$. Indeed $\mathcal{N}(z, tQ)$ and $\mathcal{N}(z_1, t\lambda_1) \otimes \mathcal{N}(z', tQ')$ have the same characteristic function and so they coincide, see Section 1.1.

Then we can easily establish that $\mathcal{N}(x_1, t\lambda_1)(A) \geq \mathcal{N}(-x_1, t\lambda_1)(A)$ for any $A \in \mathcal{B}(\mathbb{R}_+)$, $x_1 > 0$, $t > 0$, and so,

$$\mathcal{N}(x, tQ)(B) = \mathcal{N}(x_1, t\lambda_1) \otimes \mathcal{N}(x', tQ')(B) \geq \mathcal{N}(-x_1, t\lambda_1) \otimes \mathcal{N}(x', tQ')(B)$$

$$= \mathcal{N}(\phi_1(x), tQ)(B), \quad B \in \mathcal{B}(H_+), \ x \in H_+, \ t > 0,$$

consequently $\mathcal{N}(x, tQ) - \mathcal{N}(\phi_1(x), tQ)$ is a (positive) measure on $\mathcal{B}(H_+)$ and

$$\mathcal{N}(x, tQ) - \mathcal{N}(\phi_1(x), tQ) = [\mathcal{N}(x_1, t\lambda_1) - \mathcal{N}(-x_1, t\lambda_1)] \otimes \mathcal{N}(x', tQ'), \ \forall x \in H_+. \tag{5.2.5}$$

For any $f \in \mathcal{C}_b(H_+)$, $t > 0$, $x \in H_+$, we have by changing variable, see (1.1.13), since $\phi_1^* Q \phi_1 = Q$.

$$\begin{aligned} P_t f(x) &= \int_H Ef(x+y)\, \mathcal{N}(0, tQ) dy \\ &= \int_{H_+} f(y)\, \mathcal{N}(x, tQ) dy - \int_{H_-} f(\phi_1(y))\, \mathcal{N}(x, tQ) dy \\ &= \int_{H_+} f(y)\, \mathcal{N}(x, tQ) dy - \int_{H_+} f(y)\, \mathcal{N}(\phi_1(x), tQ) dy \\ &= \int_{H_+} f(y) [\mathcal{N}(x, tQ) - \mathcal{N}(\phi_1(x), tQ)] dy \\ &= \int_{H_+} f(y_1, y')\, D(x_1, t\lambda_1) \otimes \mathcal{N}(x', tQ')(dy_1, dy'). \quad ■ \end{aligned} \tag{5.2.6}$$

[3]Let A be a Borel subset of H, $\mathcal{B}(A)$ is the σ-algebra of all Borel subsets of A.

Remark 5.2.5 Formula (5.2.4) yields, by changing variable and by the Fubini Theorem,

$$
\begin{aligned}
P_t f(x) &= \int_{H_+} f(y_1, x'+y')\, D(x_1, t\lambda_1) \otimes \mathcal{N}(0', tQ')(dy_1, dy') \\
&= \int_{\mathbb{R}_+} \left(\frac{(e^{-\frac{(x_1-y_1)^2}{2t\lambda_1}} - e^{-\frac{(x_1+y_1)^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}} \right) dy_1 \int_{H'} f(y_1, x'+y')\, \mathcal{N}(0', tQ') dy'.
\end{aligned}
\tag{5.2.7}
$$

We also need the following semigroups on $\mathcal{C}_b(H_+)$:

$$
\begin{aligned}
U_t f(x) &\stackrel{\text{def}}{=} \int_{H_+} f(y_1, x'+y')\, D(x_1, t\lambda_1)(dy_1), \quad x \in H_+, \\
O'_t f(x) &\stackrel{\text{def}}{=} \int_{H_+} f(x_1, x'+y')\, \mathcal{N}(0', tQ')(dy'), \quad x \in H_+,
\end{aligned}
\tag{5.2.8}
$$

where $f \in \mathcal{C}_b(H_+)$. Proceeding as for O_t, we can check that O'_t is a strongly continuous semigroup of contractions on $\mathcal{C}_b(H_+)$, and $P_t = U_t\, O'_t$ for any $t \geq 0$. ∎

We point out that P_t and U_t are not strongly continuous semigroups on $\mathcal{C}_b(H_+)$. We only prove this fact for P_t in the following proposition.

Proposition 5.2.6 *Set* $\mathcal{C}_0(H_+) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_b(H_+)$ *such that* $f(z) = 0, \quad z \in \partial H_+\}$ *then it holds:*

$$\lim_{t\to 0^+} P_t g = g \text{ in } \mathcal{C}_b(H_+) \quad \Leftrightarrow \quad g \in \mathcal{C}_0(H_+).$$

Proof We verify $\Rightarrow$.
For any $f \in \mathcal{C}_b(H_+)$ we claim that $P_t f(z) = 0$ for any $z \in \partial H_+$, $t > 0$.
To deduce this property, fix $\hat{z} = (0, \hat{z}') \in \partial H_+$ and take a sequence $(z_n) \subset H_+$ such that $z_n \mapsto \hat{z}$. Then $\lim_{n\to\infty} P_t f(z_n) = P_t f(\hat{z})$, $t \geq 0$ and using the Dominated Convergence Theorem we obtain

$$
\begin{aligned}
P_t f(\hat{z}) &= \lim_{n\to\infty} P_t f(z_n) = \lim_{n\to\infty} \int_H E f(z_n + y)\, \mathcal{N}(0, tQ) dy \\
&= \int_H E f(\hat{z} + y) \mathcal{N}(0, tQ) dy = \int_H E f(y_1, \hat{z}' + y')\, \mathcal{N}(0, tQ) dy.
\end{aligned}
$$

Changing variable in the last integral as in (5.2.6), we find $P_t f(\hat{z}) = 0$.

Let now $g \in \mathcal{C}_b(H_+)$ such that $\lim_{t\to 0^+} P_t g = g$ in $\mathcal{C}_b(H_+)$ (or in $\mathcal{C}_b(\overline{H}_+)$). It follows that $g(z) = 0$ for any $z \in \partial H_+$.
We prove $\Leftarrow$.
Take $g \in \mathcal{C}_b(H_+)$ then it is clear that $Eg \in \mathcal{C}_b(H)$. Since $P_t g(x) = O_t E g(x)$, for any $x \in H_+$, we find $\lim_{t\to 0^+} O_t E g = E g$ in $\mathcal{C}_b(H)$. This yields that $\lim_{t\to 0^+} P_t g = g$ in $\mathcal{C}_b(H_+)$. ∎

By the last result we know that $\mathcal{C}_0(H_+)$ is the *maximal subspace* of $\mathcal{C}_b(H_+)$ on which P_t is strongly continuous. Now we state a regularity property for P_t.

Proposition 5.2.7 *Let* $f \in \mathcal{C}_b(H_+)$, *then* $P_t f \in \mathcal{C}_Q^\infty(H_+)$, $t > 0$.

Proof First we prove that for any $t > 0$, $O_t E f \in \mathcal{C}_b(H)$.
From (iii) of Lemma 5.2.1 we know that:

$$E(RO_t Ef)(x) = O_t Ef(x), \quad x \in H, \quad t \geq 0,$$

but $E(RO_t Ef) = EP_t f$ and $P_t f \in \mathcal{C}_0(H_+)$ for $t > 0$ (see the proof of Proposition 5.2.6), so $O_t Ef \in \mathcal{C}_b(H)$.

By Proposition 3.2.2, we know that

$$\text{for any } f \in \mathcal{C}_b(H) \text{ one has } O_t f \in \mathcal{C}_Q^\infty(H), \; t > 0. \tag{5.2.9}$$

Using this result and the fact that $P_t f = RO_{t/2} O_{t/2} Ef$, $t \geq 0$, we obtain the assertion. ∎

Now we associate a *generator* to the semigroup P_t on $\mathcal{C}_b(H_+)$, using a well known result of Hille about *pseudo-resolvents* (see for instance Yosida [88, §VIII.4.1]). This approach is similar to that used in Cerrai [14] in order to define a generator for the Ornstein-Uhlenbeck operators in $\mathcal{C}_b(H)$. For a further characterization of this generator we refer to Sections 6.2 and 6.3.

First we define operators $F_\lambda : \mathcal{C}_b(H_+) \to \mathcal{B}_b(H_+)$, $\lambda > 0$,

$$F_\lambda f(x) \stackrel{\text{def}}{=} \int_0^{+\infty} e^{-\lambda t} \; P_t f(x)\, dt, \quad f \in \mathcal{C}_b(H_+), \quad x \in H_+, \tag{5.2.10}$$

where the map $t \to P_t f(x)$ is continuous from $[0, \infty)$ into $\mathbb{R}$ for any $x \in H_+$ (see Proposition 5.2.3). In the following proposition we verify that the operators F_λ, $\lambda > 0$ are pseudo-resolvents on $\mathcal{C}_b(H_+)$. This means that $F_\lambda \in \mathcal{L}(\mathcal{C}_b(H))$, is one to one, for any $\lambda > 0$, and the resolvent identity holds.

Before going on we fix a notation. Let E be a Banach space, $\mathcal{Z}$ be a linear operator on E and I_E be the identity. We will use $(\lambda - \mathcal{Z})$ to denote $\lambda I_E - \mathcal{Z}$, $\lambda > 0$.

Proposition 5.2.8 *There exists a unique linear closed operator on* $\mathcal{C}_b(H_+)$, $\mathcal{T} : D(\mathcal{T}) \to \mathcal{C}_b(H_+)$ *such that setting* $R(\lambda, \mathcal{T}) = (\lambda - \mathcal{T})^{-1}$, *we have*

$$R(\lambda, \mathcal{T}) = F_\lambda \quad \textit{and} \quad \|R(\lambda, \mathcal{T})\|_{\mathcal{L}(\mathcal{C}_b(H_+))} \leq \tfrac{1}{\lambda}, \qquad \lambda > 0.$$

Proof *Step 1.* We first prove that $F_\lambda \in \mathcal{L}(\mathcal{C}_b(H_+))$, $\lambda > 0$.
To this end, we show that for any $f \in \mathcal{C}_b(H_+)$, $F_\lambda f \in \mathcal{C}_b(H_+)$. Taking into account Proposition 5.2.2, we get

$$|F_\lambda f(x) - F_\lambda f(z)| \leq \textstyle\int_0^{+\infty} e^{-\lambda t} \; |P_t f(x) - P_t f(z)|\, dt$$

$$\leq \theta_f(|x - z|) \textstyle\int_0^{+\infty} e^{-\lambda t} \; \left(1 + \frac{1}{\sqrt{t}}\right) dt \; \leq c_\lambda\, \theta_f(|x - z|), \quad x,\, z \in H_+,$$

where $\theta_f(r) \stackrel{\text{def}}{=} 2\omega_f(r) + 2\frac{\|f\|_0}{\sqrt{2\pi\lambda_1}}\, r$, $r \geq 0$. This implies the uniform continuity of $F_\lambda f$. Moreover we have

$$|F_\lambda f(x)| \le \int_0^\infty e^{-\lambda u}\, |P_u f(x)|\, du \le \frac{\|f\|_0}{\lambda}, \quad f \in \mathcal{C}_b(H),\ x \in H.$$

It follows that $F_\lambda f \in \mathcal{C}_b(H_+)$ and moreover $\|F_\lambda\|_{\mathcal{L}(\mathcal{C}_b(H_+))} \le \frac{1}{\lambda},\ \lambda > 0.$

Step 2. Now we verify the resolvent identity: $F_\lambda - F_\mu = (\mu - \lambda)\, F_\lambda\, F_\mu,$ for $\lambda,\ \mu > 0.$
We claim that for any $u \ge 0$:

$$P_u \left(\int_0^{+\infty} e^{-\lambda t}\ \ P_t f\ dt \right)(x) \ = \ \int_0^{+\infty} e^{-\lambda t}\ \ P_{t+u} f(x)\ dt, \qquad f \in \mathcal{C}_b(H_+),\ \ x \in H_+.$$

Indeed, using the integral representation of O_t, the Fubini Theorem, and (iii) of Lemma 5.2.1 we have for any $x \in H_+$,

$$P_u \left(\int_0^{+\infty} e^{-\lambda t}\ \ P_t f\ dt \right)(x) = \ O_u E \left(\int_0^{+\infty} e^{-\lambda t}\ \ R O_t E f dt \right)(x) = \int_0^{+\infty} e^{-\lambda t}\ \ P_{t+u} f(x) dt,$$

Now the resolvent identity follows readily, by changing variable and integrating by parts,

$$\begin{aligned} F_\lambda F_\mu f(x) &= \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s}\, P_{t+s} f(x)\, ds\, dt \\ &= \int_0^\infty e^{-\lambda t} \int_t^\infty e^{-\mu(u-t)}\, P_u f(x)\, du\, dt \\ &= \int_0^\infty e^{-(\lambda-\mu)t} \int_t^\infty e^{-\mu u}\, P_u f(x)\, du\, dt \\ &= \frac{1}{\mu-\lambda}\Big(- \int_0^\infty e^{-\mu u}\, P_u f(x) du + \int_0^\infty e^{-\lambda u}\, P_u f(x) du \Big) \\ &= \frac{1}{\mu-\lambda}\Big(F_\lambda f(x) - F_\mu f(x) \Big), \ \ \lambda,\ \mu > 0,\ f \in \mathcal{C}_b(H_+),\ x \in H_+. \end{aligned}$$

Step 3. We verify that F_λ is one to one. To this end note that, for any $f \in \mathcal{C}_b(H_+)$, $x \in H_+$, $\lambda > 0$,

$$\lambda F_\lambda f(x) \ = \ \int_0^\infty e^{-u}\, P_{\frac{u}{\lambda}} f(x)\, du.$$

Letting $\lambda \to \infty$ in the right-hand side of last formula, by the Dominated Convergence Theorem, we find

$$\lim_{\lambda\to\infty} \lambda F_\lambda f(x) = f(x), \ \ f \in \mathcal{C}_b(H_+),\ x \in H_+. \tag{5.2.11}$$

If there exist λ_0 and $\hat{f} \in \mathcal{C}_b(H_+)$ such that $F_{\lambda_0}\hat{f} = 0$, we derive that $F_\lambda \hat{f} = 0$, for any $\lambda > 0$, by the resolvent identity. Now (5.2.11) yields that $\hat{f} = 0$. It follows that F_λ is one to one, $\lambda > 0$.

By the Hille Theorem, previously mentioned, there exists a unique closed operator $\mathcal{T}$ on $\mathcal{C}_b(H)$ such that

$$R(\lambda, \mathcal{T}) \ = \ F_\lambda, \quad \lambda > 0. \qquad \blacksquare$$

5.2.2 Strong and strict solutions

We want to show the connection between the Dirichlet problem (5.1.1) and the equation:

$$(\lambda - \mathcal{T})\psi \;=\; g, \quad g \in \mathcal{C}_b(H_+), \quad \lambda > 0. \tag{5.2.12}$$

First we remark that the solution $\psi \;=\; R(\lambda, \mathcal{T})g$ of (5.2.12) has the property that $\psi(z) \;=\; 0$ if $z \in \partial H_+$ and so $D(\mathcal{T}) \subset \mathcal{C}_0(H_+)$.
To see this fact, fix any $z \in \partial H_+$ and take a sequence $(z_n) \subset H_+$ such that $z_n \;\rightarrow\; z$ as $n \rightarrow \infty$. By Proposition 5.2.6, we know that $P_t(z) \;=\; 0$ for any $t > 0$. Applying the Dominated Convergence Theorem, we get

$$\psi(z) = \lim_{n\to\infty} \psi(z_n) = \lim_{n\to\infty} \int_0^{+\infty} e^{-\lambda t} \;\; P_t g(z_n)\, dt \;=\; 0. \tag{5.2.13}$$

Then we recall the definition of the space $\mathcal{C}_s^2(H_+)$, see Chapters 1, 2, for more details.
$\mathcal{C}_s^2(H_+) \overset{\text{def}}{=} \{f \in \mathcal{C}_b^{1,1}(H_+)$, ([4]) having the second Hadamard derivative $\hat{D}^2 f(x)$ at each $x \in H_+$, such that $\hat{D}^2 f(\cdot)(v) \in \mathcal{C}_b(H_+, H)$, for any $v \in H\}$.
In Theorem 2.2.10 it is shown that $\mathcal{C}_s^2(H_+)$ is dense in $\mathcal{C}_b(H_+)$. On the contrary the more "natural" space $\mathcal{C}_b^2(H_+)$ ([5]) is not dense in $\mathcal{C}_b(H_+)$, when H is infinite dimensional, see Nemirovskii and Semenov [59] and Chapter 2.

Note that it holds

$$D_Q^2 f(x) \;=\; Q^{1/2} \hat{D}^2 f(x) Q^{1/2}, \quad f \in \mathcal{C}_s^2(H_+) \cap \mathcal{C}_Q^2(H_+),\; x \in H_+. \tag{5.2.14}$$

The space $\mathcal{C}_s^2(H_+)$ is used to define the following linear operator.

Definition 5.2.9

$$\begin{cases} D(\mathcal{T}_0) \;=\; \{f \in \mathcal{C}_s^2(H_+) \text{ such that } f(z) \;=\; 0 \;\; \forall z \in \partial H_+, \\ Q^{1/2} \hat{D}^2 f(\cdot) Q^{1/2} \in \mathcal{C}_b(H_+, \mathcal{L}_1(H))\}; \\ \\ \mathcal{T}_0 \;:\; D(\mathcal{T}_0) \;\rightarrow\; \mathcal{C}_b(H_+), \;\; \mathcal{T}_0 f(x) \overset{\text{def}}{=} \dfrac{1}{2} \mathrm{Tr}\,[Q \hat{D}^2 f(x)],\; f \in D(\mathcal{T}_0),\; x \in H_+, \end{cases} \tag{5.2.15}$$

where $\hat{D}^2 f$ denotes the second Hadamard derivative of f. It is easy to verify that $D(\mathcal{T}_0) \subset \mathcal{C}_Q^2(H_+)$. Considering the orthonormal basis $\{e_k\}_{k\geq 1}$ of H, fixed at the beginning, for any $f \in D(\mathcal{T}_0)$, we have

$$\mathrm{Tr}\,[Q \hat{D}^2 f(x)] \;=\; \textstyle\sum_{k=1}^{\infty} \lambda_k D_{kk} f(x), \quad x \in H_+,$$

where $D_k f$ is the partial derivative of f in the direction e_k, $k \geq 1$ and we set $D_{hk} f \;=\; D_h(D_k f)$, $h,\, k \geq 1$.

---

[4] $\mathcal{C}_b^{1,1}(H_+)$ stands for the space of all functions in $\mathcal{C}_b(H_+)$, having a Lipschitz continuous and bounded Fréchet derivative.

[5] $\mathcal{C}_b^2(H_+)$ denotes the space of all functions in $\mathcal{C}_b^1(H_+)$, having a second Fréchet derivative uniformly continuous and bounded from H_+ into $\mathcal{L}(H)$.

Now we can clarify the notion of solution for the Dirichlet problem (5.1.1) having the initial datum $f \in \mathcal{C}_b(H_+)$.

Let $\psi \in \mathcal{C}_b(H_+)$, ψ is said to be a *strict solution* if it belongs to $D(\mathcal{T}_0)$ and it solves the equation (5.1.1); ψ is said to be a *strong solution* if there exists a sequence $(\psi_n) \subset D(\mathcal{T}_0)$ such that

$$\lim_{n\to\infty} \psi_n \;=\; \psi, \quad \lim_{n\to\infty} \Big[\lambda\psi_n \;-\; \mathcal{T}_0\psi_n\Big] \;=\; f \;\text{ in }\; \mathcal{C}_b(H_+). \;\blacksquare \tag{5.2.16}$$

To prove existence and uniqueness for strong and strict solutions, we need some preliminary results.

Lemma 5.2.10 *If $g \in \mathcal{C}_s^2(H_+) \cap \mathcal{C}_0(H_+)$ then it holds:*

(i) $Eg \in \mathcal{C}_b^{1,1}(H)$;
(ii) *if $k > 1$ there exists $D_{kk}Eg \in \mathcal{C}_b(H)$, there exists $D_{11}Eg(x)$ for any $x \notin \partial H_+$, $D_{11}Eg \in \mathcal{C}_b(H_+) \cap \mathcal{C}_b(H_-)$;*
(iii) $O_tED_{kk}g(x) \;=\; D_{kk}O_tEg(x), \quad x \in H, \;\; k \geq 1, \;\; t > 0$;
(iv) $O_tEg \in \mathcal{C}_s^2(H)$, $t > 0$, *and for the Fréchet derivative and the second Hadamard derivative of O_tEg we have:* $\|DO_tEg\|_0 \leq \|Dg\|_0$, $\|\hat{D}^2O_tEg\|_0 \;\leq\; \|\hat{D}^2g\|_0, \quad t > 0.$

Proof **(i)** Since $D(\mathcal{T}_0) \subset \mathcal{C}_0(H_+)$, see (5.2.13) it is easy to verify that $Eg \in \mathcal{C}_b(H)$. By the definition of Eg, there exists the Fréchet derivative $DEg(y)$ for $y \notin \partial H_+$ and further $DEg \in \mathcal{C}_b(H_+, H) \cap \mathcal{C}_b(H_-, H)$. Note that, for any $y \in H_-$,

$$DEg(y) \;=\; D[-g(\phi_1(y)] \;=\; -Dg[\phi_1(y)] \circ \phi_1 \;=\; -\phi_1[Dg(\phi_1(y))].$$

Let now $z \in \partial H_+$, we check that there exists the Gâteaux derivative: $D_GEg(z)$ and moreover $D_GEg(z) \;=\; Dg(z)$ (we have extended Dg to $\overline{H_+}$ for its uniform continuity).

We consider, for any $v \in H_+$,

$$\frac{Eg(z+sv) - Eg(z)}{s} \;=\; \frac{g(z+sv) - g(z)}{s} \quad s \in (0,1]. \tag{5.2.17}$$

Using the Mean Value Theorem, for any $s \in (0,1]$, there exists $\hat{s} > 0$ such that:

$$\frac{Eg(z+sv) - Eg(z)}{s} \;= < Dg(z+\hat{s}v), v > .$$

Letting $s \;\to 0^+$ yields

$$\lim_{s\to O^+} \frac{Eg(z+sv) - Eg(z)}{s} =< Dg(z), v >= \; D_1g(z)v_1,$$

since $g(0,x') \;= 0$, for any $x' \in H'$. If $v \in H_-$, by the same argument, we get

$$\lim_{s\to 0^+} \frac{Eg(z+sv) - Eg(z)}{s} \;= \lim_{s\to 0^+} \frac{-g(z+s\phi_1(v))}{s}$$
$$= - < Dg(z), \phi_1(v) > = D_1g(z)v_1 \;=< Dg(z), v > .$$

Finally if $v \in \partial H_+$, we find that $\lim_{s\to 0^+} \frac{Eg(z+sv)-Eg(z)}{s} = 0 =< Dg(z), v >$.

Thus we know that Eg has the Gâteaux derivative: $D_G Eg(y)$ at every $y \in H$. We can also prove that $D_G Eg \in \mathcal{C}_b(H,H)$. Indeed for any $z \in \partial H_+$, it holds

$$\begin{aligned} \lim_{y\to z,\, y\in H_-} D_G Eg(y) \;=\; \lim_{y\to z,\, y\in H_-} -\phi_1[Dg(\phi_1(y))] \;=\; \lim_{y\to z,\, y\in H_+} -\phi_1(Dg(y)) \\ =\; D_1 g(z) \;=\; \lim_{y\to z,\, y\in H_+} D_G g(y). \end{aligned}$$

Now for a well known result about differentiability, we can conclude that Eg is also Fréchet differentiable on H with Fréchet derivative $DEg(y) = D_G Eg(y)$, $y \in H$.

Moreover it is straightforward to verify that we also have

$$DEg \in \mathcal{C}_b^{0,1}(H,H),$$

since by Hypothesis: $DEg \in \mathcal{C}_b^{0,1}(H_+,H) \cap \mathcal{C}_b^{0,1}(H_-,H)$.

(ii) Checking all possible cases: $y \in H_+,\ y \in H_-,\ y \in \partial H_+$ we find that there exist

$$\begin{aligned} D_{kk}Eg(y) \;&=\; ED_{kk}g(y), \quad \text{for any } y \in H, \quad k > 1 \text{ and} \qquad (5.2.18) \\ D_{11}Eg(y) &=\; ED_{11}g(y), \ \text{ only if } y \notin \partial H_+. \end{aligned}$$

We extend $D_{11}Eg$ to the whole space H, setting $D_{11}Eg(z) = D_{11}g(z),\ z \in \partial H_+$, so that $D_{11}Eg \in \mathcal{B}_b(H)$.

(iii) If $k > 1$, applying (ii) , it is clear that for any $t \ge 0$:

$$O_t ED_{kk}g(y) \;=\; O_t D_{kk}Eg(y) \;=\; D_{kk}O_t Eg(y), \quad y \in H.$$

It remains to consider $k = 1$. Since $\mathcal{N}(0,tQ)\,(\partial H_+) = 0,\ t > 0$, and $D_1 Eg$ is a Lipschitz continuous map, applying the Dominated Convergence Theorem we get

$$\begin{aligned} O_t D_{11}Eg(x) \;&=\; D_1 O_t[D_1 Eg](x) \\ = \lim_{u\to 0} \int_H \frac{D_1 Eg(x+y+ue_1) - D_1 Eg(x+y)}{u} &\,\mathcal{N}(0,tQ)dy \\ = \int_H D_{11}Eg(x+y)\,\mathcal{N}(0,tQ)dy \;&=\; O_t D_{11}Eg(x), \quad x \in H,\ t > 0. \end{aligned}$$

(iv) Fix $t > 0$. It is easy to prove that $O_t Eg \in \mathcal{C}_b^{1,1}(H)$, see (2.2.9), and moreover that it holds:

$$< DO_t Eg(x), u > = \; O_t(< DEg(\cdot), u >)(x), \quad x \in H,\ u \in H.$$

Consequently the first estimate of (iv) follows.

It is a straightforward computation to check that $DEg : H \to H$ is Hadamard differentiable at any point $y \notin \partial H_+$. Indeed we have

$$\hat{D}^2 Eg(y) \;=\; \hat{D}^2 g(y),\ y \in H_+, \quad \hat{D}^2 Eg(x) \;=\; -\phi_1 \circ \hat{D}^2 g(\phi_1(x)) \circ \phi_1,\ \ x \in H_-.$$

Moreover for any $v \in H$, $\hat{D}^2 Eg(\cdot)(v) \in \mathcal{C}_b(H_+, H) \cap \mathcal{C}_b(H_-, H)$. We extend $\hat{D}^2 Eg$ to the whole space H, setting $\hat{D}^2 Eg(z) = 0$, for any $z \in \partial H_+$, so that for each $v \in H$, we have that $\hat{D}^2 Eg(\cdot)(v)$ is a Borel bounded map from H into H.

Now we prove that $O_t Eg$ has the second Hadamard derivative on H and that it holds:

$$\hat{D}^2 O_t Eg(x)(v) \;=\; \int_H \hat{D}^2 Eg(x+y)(v)\, \mathcal{N}(0, tQ) dy, \quad x,\, v \in H, \tag{5.2.19}$$

where the integral is in Bochner's sense. Fix $x \in H$ and a compact set K in H and consider the mapping $\Lambda : H \times H \times (0,1] \to H$,

$\Lambda(y, v, s) \overset{\text{def}}{=} \frac{DEg(x+y+sv) - DEg(x+y)}{s} - \hat{D}^2 Eg(x+y)(v), \quad y,\, v \in H,\; s \in (0,1]$.

Now (5.2.19) follows by showing that

$$\lim_{s \to 0^+} \sup_{v \in K} \int_H |\Lambda(y, v, s)|\, \mathcal{N}(0, tQ) dy \;=\; 0. \tag{5.2.20}$$

Let L be a countable dense set in K. For any $y \in H$, $s \in (0,1]$ the map $\Lambda(y, \cdot, s) \in \mathcal{C}_b(H_+, H) \cap \mathcal{C}_b(H_-, H)$ and so the following property holds:

$$\sup_{v \in K} |\Lambda(y, v, s)| \;=\; \sup_{v \in L} |\Lambda(y, v, s)|, \quad y \in H,\; s \in (0,1].$$

We point out that for any fixed $s \in (0,1]$, the map $\sup_{v \in L} |\Lambda(\cdot, v, s)|$ is a real Borel map on H. Further we have:

$$\sup_{s \in (0,1]} \sup_{y \in H} \sup_{v \in L} |\Lambda(y, v, s)| \le 2C \|\hat{D}^2 g\|_0, \tag{5.2.21}$$

where we have choosen C such that for any $v \in K$, $|v| \le C$. Using the estimate (5.2.21) and applying the Dominated Convergence Theorem we get (5.2.20) and so (5.2.19). Using (5.2.19) it is simple to verify that for any $v \in H$, $\hat{D}^2 O_t Eg(\cdot)(v) \in \mathcal{C}_b(H, H)$ and that the second estimate of (iv) holds. The proof is complete. ∎

In the following theorem we identify $\mathcal{C}_b(H')$ with the subspace of $\mathcal{C}_b(H_+)$ of all functions which are constant in the first variable (see formula (5.1.3)).

Theorem 5.2.11 *For any* $g \in \mathcal{C}^2_s(H_+)$ *then* $R(\lambda, \mathcal{T})g \in D(\mathcal{T}_0)$, $\lambda > 0$.

Proof Fix $g \in \mathcal{C}^2_s(H_+)$, $\lambda > 0$ and set $\psi = R(\lambda, \mathcal{T})g$. It is clear that $\psi(0, x') = 0$, $x' \in H'$, using formula (5.3.27). We set for any $x = (x_1, x') \in H_+$, $g(x) = f(x) + h(x')$,
where $f(x) = g(x) - g(0, x')$, $f \in \mathcal{C}_0(H_+)$, and $h(x') = f(0, x')$, $h \in \mathcal{C}_b(H')$.
Let now

$$\psi_1 \;=\; R(\lambda, \mathcal{T})f, \quad \psi_2 \;=\; R(\lambda, \mathcal{T})h, \;\text{ so that }\; \psi \;=\; \psi_1 + \psi_2.$$

We split up the proof into two parts.
(a) First we prove that $\psi_1 \in D(\mathcal{T}_0)$.
According to (iv) of Lemma 5.2.10 we know that $P_t f \in \mathcal{C}^2_s(H_+)$, $t \ge 0$ and we have:

$$\|DP_tf(x)\|_H \;\le\; \|Df\|_0 \;\le\; \|Dg\|_0, \quad \text{for} \quad t\ge 0, \quad x\in H_+ \quad \text{and}$$

$$\|\hat D^2 P_tf(x)\|_{\mathcal{L}(H)} \;\le\; \|\hat D^2 f\|_0 \;\le\; \|\hat D^2 g\|_0, \quad \text{for} \quad t\ge 0, \;\; x\in H_+.$$

where "D " denotes the Fréchet derivative and "$\hat D^2$" the second Hadamard derivative. In view of these estimates we can differentiate under the integral defining ψ_1 and get easily that $\psi_1 \in \mathcal{C}^2_s(H_+)$, see also the proof of (iv) in Lemma 5.2.10.

It remains to prove that $Q^{1/2}\hat D^2\psi_1 Q^{1/2} \in \mathcal{C}_b(H_+,\mathcal{L}_1(H))$. The boundedness of $Q^{1/2}\hat D^2\psi_1 Q^{1/2}$ is clear, since

$$\|Q^{1/2}\hat D^2\psi_1 Q^{1/2}\|_{0,\mathcal{L}_1} \le \|Q^{1/2}\|^2_{\mathcal{L}_2(H)}\,\|\hat D^2\psi_1\|_{0,\mathcal{L}(H)}.$$

We verify the uniform continuity. We have the following formula, for any $x\in H_+$,

$$Q^{1/2}\hat D^2\psi_1(x)Q^{1/2}\,(v) \;=\; \int_0^\infty e^{-\lambda t} Q^{1/2}\hat D^2[O_tEf](x)Q^{1/2}\,(v)\,dt, \;\; v\in H, \qquad (5.2.22)$$

since, by (5.2.19), for any $x,\ v\in H$, the map $[0,\infty)\to H,\;\; t\mapsto \hat D^2 O_tEf(x)\,(v)$ is continuous.

Now we consider $Q^{1/2}(\hat D^2 O_tEf)Q^{1/2}$. Recalling (iv) of Lemma 5.2.10 and formula (5.2.9), we know that $O_tEf \in \mathcal{C}^2_s(H)\cap\mathcal{C}^\infty_Q(H),\;\; t>0$. Moreover it is easy to check that $Q^{1/2}(\hat D^2 O_tEf)Q^{1/2} \;=\; D^2_Q O_tEf$.
Then we apply the following result, see Proposition 3.3.3 and (3.3.23): let $\varphi\in\mathcal{C}^1_b(H)$, we have that $D^2_Q O_t\varphi(x)\in\mathcal{L}_1(H)$, for any $x\in H$ and moreover for any $t>0$,

$$D^2_Q O_t\varphi \in \mathcal{C}_b(H,\mathcal{L}_1(H)), \quad \|D^2_Q O_t\varphi(x) - D^2_Q O_t\varphi(z)\|_{\mathcal{L}_1(H)} \le c\frac{1}{\sqrt t}\,\omega_{D\varphi}(|x-z|), \qquad (5.2.23)$$

where $x,\ z\in H,\;\; \omega_{D\varphi}(\cdot)$ denotes the modulus of continuity of $D\varphi$ and $c=c(Q)$.

Using the last estimate with φ replaced by Ef, we get for any $N\in\mathcal{L}(H)$ of finite rank and such that $\|N\|_{\mathcal{L}(H)}\le 1$ (see also the proof of Theorem 4.2.2),

$$\begin{aligned}
&|\mathrm{Tr}\,(N[D^2_Q\psi_1(x) - D^2_Q\psi_1(z)])|\\
&\le \int_0^\infty e^{-\lambda u}\,|\mathrm{Tr}\,(N[D^2_Q O_tEf(x) - D^2_Q O_tEf(z)])|\,du \qquad\qquad (5.2.24)\\
&\le c\,\omega_{DEf}(|x-z|)\int_0^\infty e^{-\lambda u}\frac{1}{\sqrt u}du = c\sqrt{\frac{\pi}{\lambda}}\,\omega_{DEf}(|x-z|),
\end{aligned}$$

where $x,\ z\in H_+$. Taking the supremum over all N and invoking Lemma 1.1.3, we derive

$$\|D^2_Q\psi_1(x) - \, D^2_Q\psi_1(z)\|_{\mathcal{L}_1(H)} \;\le C\,\omega_{Df}(|x-z|),\;\; x,\,z\in H_+$$

and the uniform continuity of $Q^{1/2}D^2\psi_1Q^{1/2}$ follows.

(b) Now we study the regularity of ψ_2.
We have:

$$P_th(x) \;=\; U_t\;O'_th(x) \;=\; \eta_t(x_1)\;O'_th(x'), \quad x\in H_+,\; t>0,$$

where the semigroups U_t and O'_t are defined in Remark 5.2.5 and

$$\eta_t(x_1) \stackrel{\text{def}}{=} U_t 1(x_1) \;=\; \int_0^\infty \frac{(e^{-\frac{(x_1-y_1)^2}{2t\lambda_1}} \;-\; e^{-\frac{(x_1+y_1)^2}{2t\lambda_1}})}{\sqrt{2\pi t\lambda_1}}\, dy_1 \;=\; \int_0^{x_1} \frac{2e^{-\frac{u^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}}\, du. \quad (5.2.25)$$

We write $O'_t h(x')$ instead of $O'_t h(x)$, since $O'_t h$ is constant in the first variable. We can compute

$$D_1\eta_t(x_1) \;=\; \frac{2}{\sqrt{2\pi t\lambda_1}} e^{-x_1^2/2t\lambda_1}, \;\; D_{11}\eta_t(x_1) \;=\; -\frac{2}{\sqrt{2\pi t\lambda_1}}\frac{x_1}{t\lambda_1} e^{-x_1^2/2t\lambda_1}. \qquad (5.2.26)$$

We know that:

$$\psi_2(x) \;=\; \int_0^{+\infty} e^{-\lambda t} \;\; [\eta_t O'_t h]\,(x)\; dt, \quad x \in H_+.$$

To get differentiability of ψ_2, we need to differentiate under the integral; to this end, estimates for the derivatives of $\eta_t O'_t h$ are necessary. We denote the first Fréchet in the variable $x' \in H'$, by D'. As concerns the Fréchet derivative one has, for any $v \in H$,

$$< D(\eta_t O'_t h)(x), v >= \; D_1\eta_t(x_1)v_1 \; O'_t h(x') + \eta_t(x_1) \; < D' O'_t h(x'), v' >, \;\; x \in H_+,$$

$$\text{so that} \qquad \|D(\eta_t O'_t h)\|_0 \;\le\; c\,(1 + \frac{1}{\sqrt{t}})\;(\|h\|_0 + \|Dh\|_0), \qquad t > 0.$$

By the last estimate, we derive easily that $\psi_2 \in \mathcal{C}^1_b(H)$. Now we deal with the second Hadamard derivative $\hat{D}^2$. For any $u, v \in H,\; x \in H_+,\; t > 0,$ we set

$$< \hat{D}^2(\eta_t O'_t h)(x)\;(v), u > \,=\, < [J_1(x,t) + J_2(x,t) + J_3(x,t)]\, v, u >, \qquad (5.2.27)$$

where $\quad J_1(x,t),\; J_2(x,t),\; J_3(x,t) \in \mathcal{L}(H), \;\; x \in H_+,\, t > 0,$

$$\begin{aligned} &< J_1(x,t)v, u > \;\stackrel{\text{def}}{=}\; D_{11}\eta_t(x_1)v_1u_1 O'_t h(x'),\\ &< J_2(x,t)v, u > \stackrel{\text{def}}{=} D_1\eta_t(x_1)v_1 < D' O'_t h(x'), u' > + D_1\eta_t(x_1)u_1 < D' O'_t h(x'), v' >,\\ &< J_3(x,t)v, u > \;\stackrel{\text{def}}{=}\; \eta_t(x_1) < \hat{D}'^2 O'_t h(x')(v'), u' > . \end{aligned}$$

As concerns $J_2 + J_3$, we get easily, for any $x \in H_+,\; u, v \in H$,

$$\sup_{x\in H_+} \,|< [J_2(x,t) + J_3(x,t)]\,(u), v >| \le\; \tilde{c}|u||v|(1 + \frac{1}{\sqrt{t}})(\|Dh\|_0 + \|\hat{D}^2 h\|_0), \; t > 0.$$

We consider now the more difficult term J_1. We find for any $t > 0,\; \delta > 0$:

$$\sup_{x\in H_+^\delta} \,\|D_{11}\eta_t O'_t h\|_0 \;\le\; \sup_{x_1>\delta} \,|\frac{2}{\sqrt{2\pi t\lambda_1}}\frac{x_1}{t\lambda_1} e^{-x_1^2/2t\lambda_1}|\;\|h\|_0 \;\le\; \frac{c}{\sqrt{t}\;\delta}\|h\|_0, \qquad (5.2.28)$$

where $H_+^\delta \stackrel{\text{def}}{=} \{(x_1, x') \in H_+ \;/\; x_1 > \delta\}$, $\quad$ with $\delta > 0$. By the above estimate we can only conclude that the second Hadamard derivative $\hat{D}^2\psi_2(x)$ does exist for any

$x \in H_+$ and moreover that $\psi_2 \in \mathcal{C}_s^2(H_+^\delta)$, for any $\delta > 0$. To get that $\psi_2 \in \mathcal{C}_s^2(H_+)$ we need to handle the following integral:

$$I(x) \stackrel{\text{def}}{=} -\frac{2}{\sqrt{2\pi\lambda_1}} \int_0^\infty \frac{x_1}{\lambda_1 t\sqrt{t}} e^{\frac{-x_1^2}{2t\lambda_1}}\, e^{-\lambda t}\, O_t' h(x')\, dt, \tag{5.2.29}$$

and to prove that $I \in \mathcal{C}_b(H_+)$. Putting for any $x_1,\ t > 0$,

$$\begin{aligned} \frac{x_1}{\sqrt{t}} &= u, \quad \frac{-x_1}{2t\sqrt{t}}\, dt \;=\; du, \qquad \text{we have} \\ I(x) &= -\frac{4}{\sqrt{2\pi\lambda_1}\lambda_1} \int_0^\infty e^{-\frac{u^2}{2\lambda_1}}\, e^{-\lambda\,\frac{x_1^2}{u^2}}\;\; O'_{\frac{x_1^2}{u^2}} h(x')\, du, \quad x \in H_+. \end{aligned} \tag{5.2.30}$$

Now the boundedness follows since for any $x \in H_+$,

$$|I(x)| \le c_1 \int_0^\infty e^{-\frac{u^2}{2\lambda_1}} \left| O'_{\frac{x_1^2}{u^2}} h(x')\right| du \;\le\; \hat{c}_{\lambda_1}\, \|h\|_0.$$

Let us prove the uniform continuity of I. For any $z_1 > 0$, $\;z',\, x' \in H'$ we have

$$\begin{aligned} |I(z_1, z') - I(z_1, x')| \;&\le\; \frac{4}{\sqrt{2\pi\lambda_1}\lambda_1} \int_0^\infty e^{-\frac{u^2}{2\lambda_1}}\, e^{-\lambda\,\frac{x_1^2}{u^2}} \left| O'_{\frac{z_1^2}{u^2}} h(z') - \; O'_{\frac{z_1^2}{u^2}} h(x')\right| du \\ &\le\; c_{\lambda_1}\, \omega_h(|x' - z'|), \end{aligned} \tag{5.2.31}$$

that yields the uniform continuity of $I(z_1, \cdot)$ with the modulus of continuity independent of z_1. To get the uniform continuity of I, we prove that also $I(\cdot, x')$ is uniformly continuous with the modulus of continuity independent of x'.

To this end take any sequence $(s_n) \subset \mathbb{R}_+$ with $s_n \to 0$. We have to prove that

$$\lim_{n\to\infty} \sup_{w' \in H'} \sup_{x_1 \in \mathbb{R}_+} |I(x_1, w') - I(x_1 + s_n, w')| \;=\; 0. \tag{5.2.32}$$

We can write

$$\begin{aligned} &\sup_{(x_1,w')\in\mathbb{R}_+\times H'} |I(x_1, w') - I(x_1 + s_n, w')| \\ &\le\; \frac{4}{\sqrt{2\pi\lambda_1}\lambda_1} \sup_{(x_1,w')\in\mathbb{R}_+\times H'} \int_0^\infty e^{-\frac{u^2}{2\lambda_1}} \left| e^{-\lambda\frac{(x_1+s_n)^2}{u^2}}\; O'_{\frac{(x_1+s_n)^2}{u^2}} h(w') - e^{-\lambda\,\frac{(x_1)^2}{u^2}}\; O'_{\frac{x_1^2}{u^2}} h(w')\right| du \\ &\le\; c_{\lambda_1} \sup_{(x_1,w')\in\mathbb{R}_+\times H'} \int_0^\infty e^{-\frac{u^2}{2\lambda_1}}\, e^{-\lambda\,\frac{(x_1+s_n)^2}{u^2}} \left| O'_{\frac{(x_1+s_n)^2}{u^2}} h(w') \;-\; O'_{\frac{(x_1)^2}{u^2}} h(w')\right| du \\ &+ c_{\lambda_1} \sup_{(x_1,w')\in\mathbb{R}_+\times H'} \int_0^\infty e^{-\frac{u^2}{2\lambda_1}} \left| e^{-\lambda\,\frac{(x_1+s_n)^2}{u^2}} \;-\; e^{-\lambda\,\frac{(x_1)^2}{u^2}}\right| O'_{\frac{x_1^2}{u^2}} h(w')\, du \;=\; \Gamma_1(n) \;+\; \Gamma_2(n). \end{aligned} \tag{5.2.33}$$

Fix a dense countable subset D_+ in $[0,\infty)\times H'$ and define the map

$$T(n,u):\mathbb{N}\times\mathbb{R}_+\to\mathbb{R},\quad T(n,u)\stackrel{\text{def}}{=}\sup_{(x_1,w')\in\mathbb{R}_+\times H'}\left|O'_{\frac{(x_1+s_n)^2}{u^2}}h(w')\ -O'_{\frac{(x_1)^2}{u^2}}h(w')\right|;$$

$$\begin{aligned}&\text{There results}\quad T(n,u)=\sup_{(x_1,w')\in D_+}\left|O'_{\frac{(x_1+s_n)^2}{u^2}}h(w')-O'_{\frac{(x_1)^2}{u^2}}h(w')\right|,\\&\text{since}\quad O'_{\frac{(\cdot)^2}{u^2}}h(\cdot)\in\mathcal{C}_b(\mathbb{R}_+\times H').\end{aligned}\tag{5.2.34}$$

We obtain, for any $n\ge 1$, $u>0$,

$$\begin{aligned}T(n,u)\ &\le\sup_{(x_1,w')\in D_+}\int_H|h(w'+\frac{x_1+s_n}{u}y')-h(w'+\frac{x_1}{u}y')|\,\mathcal{N}(0,Q')dy'\\&\le\int_H\omega_h(\frac{s_n}{u}y')\,\mathcal{N}(0,Q')dy'.\end{aligned}$$

This yields, applying the Dominated Convergence Theorem,

$$\lim_{n\to\infty}T(n,u)\ =\ 0,\quad u>0.$$

Since D_+ is countable, $T(n,\cdot)$ is a Borel function on $\mathbb{R}_+$, for any $n\ge 1$. Moreover we have $T(n,u)\le\ 2\|h\|_0\ \ n\ge 1,\ \ u>0$. Hence, invoking again the Dominated Convergence Theorem, we infer

$$\lim_{n\to\infty}\Gamma_1(n)\ \ \le\ \lim_{n\to\infty}c_{\lambda_1}\ \int_0^\infty e^{-\frac{u^2}{2\lambda_1}}\,T(n,u)\,du\ =\ 0.$$

As concerns Γ_2, we find

$$\Gamma_2(n)\le\|h\|_0c_{\lambda_1}\sup_{(x_1,w')\in\mathbb{R}_+\times H'}\int_0^\infty e^{-\frac{u^2}{2\lambda_1}}\left|e^{-\lambda\,\frac{(x_1+s_n)^2}{u^2}}\ -e^{-\lambda\,\frac{(x_1)^2}{u^2}}\right|du$$

Proceeding as for Γ_1, we deduce that $\lim_{n\to\infty}\Gamma_2(n)=0$. It follows that $I\in\mathcal{C}_b(H_+)$ and consequently $\psi_2\in\mathcal{C}^2_s(H_+)$.

It remains to check that $Q^{1/2}\hat{D}^2\psi_2Q^{1/2}\in\mathcal{C}_b(H_+,\mathcal{L}_1(H))$. To this purpose, fix $t>0$ and consider, using formula (5.2.27),
$Q^{1/2}<\hat{D}^2(\eta_tO'_th)(x)Q^{1/2}=\ J_1(x,t)Q+Q^{1/2}J_2(x,t)Q^{1/2}+Q^{1/2}J_3(x,t)Q^{1/2},\ \ x\in H_+$.

It is clear that the map

$$x\mapsto\int_0^\infty e^{-\lambda t}J_1(x,t)\,Q\,dt\ \text{ belongs to }\ \mathcal{C}_b(H_+,\mathcal{L}_1(H)).$$

so we deal with $Q^{1/2}J_2Q^{1/2}$ and $Q^{1/2}J_3Q^{1/2}$. We have that for any $x\in H_+,\ t>0:$

$$\|J_2(x,t)\|_{\mathcal{L}(H)}\le\ 2|D_1\eta_t(x_1)|\,\|D'O'_th(x')\|_H\ \le\ 2\frac{c}{\sqrt{t}}\,\|Dh\|_0$$

and $J_2(\cdot,t)\in\mathcal{C}_b(H_+,\mathcal{L}(H))$, for any $t>0$, with the modulus of continuity

$$\omega_{J_2(\cdot,t)}(s)\ \le\ 2\|Dh\|_0\,\omega_{D_1\eta_t}(s)\ +\ 2\frac{c}{\sqrt{t}}\,\omega_{Dh}(s),\ \ t>0,\ s\ge 0.$$

It follows that the map

$$x\mapsto\int_0^\infty e^{-\lambda t}\,Q^{1/2}J_2(x,t)Q^{1/2}\,dt\ \text{ belongs to }\ \mathcal{C}_b(H_+,\mathcal{L}_1(H)).$$

In order to estimate $Q^{1/2}J_3Q^{1/2}$ we can use formula (5.2.23) with H and O_t replaced by H' and O'_t, for any $x',\ z'\in H',\ \ t>0$ it holds:
$\|Q^{1/2}\,D'^2O'_th(x')\,Q^{1/2}\ -\ Q^{1/2}\,D'^2O'_th(z')\,Q^{1/2}\|_{\mathcal{L}_1(H)}\ \le\ C\frac{1}{\sqrt{t}}\,\omega_{Dh}(\|x'-z'\|_{H'})$.
Then arguing as for $Q^{1/2}\hat{D}^2\psi_1Q^{1/2}$, see (5.2.24), we obtain easily the uniform continuity of $Q^{1/2}\hat{D}^2\psi_2Q^{1/2}$. The proof is complete. ∎

From the previous proof we can deduce the following result, recalling that we identify $\mathcal{C}_b(H')$ with a subspace of $\mathcal{C}_b(H_+)$ (see formula (5.1.3)).

Corollary 5.2.12 *Let $h\in\mathcal{C}_b(H')$ and $\psi\ =\ R(\lambda,\mathcal{T})h,\ \ \lambda>0$. Then there exists $D_{11}\psi\in\mathcal{C}_b(H_+)$ and a constant $c=c(\lambda,Q)$ such that: $\|D_{11}\psi\|_0\le\ c\|h\|_0$.*

The next theorem clarify that $D(\mathcal{T}_0)$ is a *core* ([6]) for $\mathcal{T}$.

Theorem 5.2.13 *It holds:*
(i) $D(\mathcal{T}_0)\subset D(\mathcal{T})$ and $\mathcal{T}$ extends $\mathcal{T}_0$,
(ii) $D(\mathcal{T}_0)$ is dense in $D(\mathcal{T})$ with respect to the graph norm.

Proof (i) Let $f\in D(\mathcal{T}_0)$, we first remark that $P_tf\in D(\mathcal{T}_0),\ \ t>0$. This fact follows easily combining (iv) of Lemma 5.2.10 with formulas (5.2.23) and (5.2.14).

Now we split up the proof into several steps.

(a) $P_t\mathcal{T}_0f\ =\ \mathcal{T}_0P_tf,\ \ \ t\ge 0$.
For $x\in H_+,\ \ t\ge 0$, using formula (5.2.18) and (iii) of Lemma 5.2.10 and applying the Dominated Convergence Theorem, we find

$$2P_t\mathcal{T}_0f(x)=\ 2O_tE\mathcal{T}_0f(x)=O_t[\ \textstyle\sum_{k=1}^\infty\lambda_kED_{kk}f\ ](x)\ =\ \sum_{k=1}^\infty\lambda_kO_t(ED_{kk}f)\,(x)$$

$$=\ \textstyle\sum_{k=1}^\infty\lambda_kD_{kk}O_tEf(x)\ =\ 2\mathcal{T}_0P_tf(x).$$

Note that $\sup_{x\in H_+}|\sum_{k=1}^n\lambda_kD_{kk}f(x)|\ \le\sup_{x\in H_+}\|QD^2f(x)\|_{\mathcal{L}_1(H)},\ \ n\ge 1$.
(b) $\frac{d}{dt}\,P_tf(x)\ =\ \mathcal{T}_0P_tf(x),\ \ \ t>0,\ \ \ x\in H_+$.
We need the following simple formula, that is a particular case of formula (3.2.5). For any $g\in\mathcal{C}_b(H)$, it holds

$$D_kO_tg(x)=\frac{1}{\lambda_kt}\int_Hy_kg(x+y)\mathcal{N}(0,tQ)dy,\ \ x\in H,\ k\ge 1,\ t>0.$$

[6]Let $(E,\|\cdot\|_E)$ be a Banach space and $\mathcal{Z}:D(\mathcal{Z})\subset E\ \to E$ be a closed operator. A subset $D\subset D(\mathcal{Z})$ is said to be a core for $\mathcal{Z}$ if it is dense in $D(\mathcal{Z})$ with respect to the graph norm.

Thus taking into account that $Ef \in \mathcal{C}_b^1(H)$, using the last formula and the Dominated Convergence Theorem, we can compute

$$\frac{d}{dt}O_tEf(x) \;=\; \frac{d}{dt}\Big(\int_H Ef(x+\sqrt{t}y)\mathcal{N}(0,Q)dy\Big)$$

$$= \frac{1}{2t}\int_H < DEf(x+y), y > \;\mathcal{N}(0,tQ)dy$$

$$= \;\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k D_{kk}O_tEf(x) \;=\; \mathcal{T}_0P_tf(x), \quad x \in H_+, \;\; t>0.$$

We have used that $|\sum_{k=1}^{\infty} D_kEf(x+y)y_k| \;\le \|DEf\|_0\,|y|, \;\; y \in H, \; x \in H_+$ and formula (1.1.12). This way the assertion is proved.

(c) $R(\lambda,\mathcal{T})(\lambda - \mathcal{T}_0)f(x) \;=\; f(x),$ for $x \in H_+, \;\; \lambda > 0$.
This implies that $f \in D(\mathcal{T})$ and $\mathcal{T}_0 f \;=\; \mathcal{T}f$. Let us notice that $(\lambda - \mathcal{T}_0)f \in \mathcal{C}_b(H_+)$. Integrating by parts and applying (b), we get

$$\int_0^{+\infty} e^{-\lambda t}\;\; P_t\mathcal{T}_0f(x)\;dt \;=\; \int_0^{+\infty} e^{-\lambda t}\;\; \frac{d}{dt}P_tf(x)\;dt$$
$$= \lambda\int_0^{+\infty} e^{-\lambda t}\;\; P_tf(x)\;dt \;-\; f(x),$$

where we have used that $\lim_{t\to 0^+} P_tf(x) \;=\; f(x), \;\; x \in H_+$ (see Proposition 5.2.3). Thus for any $x \in H_+$, we conclude that

$$R(\lambda,\mathcal{T})\;(\lambda - \mathcal{T}_0)f(x) \;=\; \int_0^{+\infty} e^{-\lambda t}\;\; P_t[\lambda f \;-\; \mathcal{T}_0 f](x)dt \;=\; f(x).$$

(ii) Let $\psi \in D(\mathcal{T})$. We want to prove that there exists $(\psi_n)_{n\ge 1} \subset D(\mathcal{T}_0)$ such that:

$$\psi_n \;\to\; \psi, \quad \mathcal{T}\psi_n \;\to\; \mathcal{T}\psi \;\text{ in }\; \mathcal{C}_b(H_+) \;\text{ as } n\to\infty. \tag{5.2.35}$$

Fix $\lambda > 0$ and set $(\lambda - \mathcal{T})\psi \;=\; g \;\in \mathcal{C}_b(H_+)$.
It is possible to choose a sequence $(g_n) \subset \mathcal{C}_s^2(H_+)$ (see Theorem 2.2.10) such that $g_n \;\to\; g$ in $\mathcal{C}_b(H_+)$ as $n\to\infty$. Set $\psi_n \;=\; R(\lambda,\mathcal{T})g_n$, then $\psi_n \in D(\mathcal{T}_0)$, for $n \ge 1$, by Theorem 5.2.11.

$R(\lambda,\mathcal{T})$ is a continuous operator on $\mathcal{C}_b(H_+)$, so that $\psi_n \;\to\; \psi$ in $\mathcal{C}_b(H_+)$ as $n\to\infty$. Now, using (i), we get $\lambda\psi_n \;-\; \mathcal{T}_0\psi_n \;=\; g_n, \quad n \ge 1$ and letting $n \;\to\; \infty$, we find $\mathcal{T}_0\psi_n \;\to\; \mathcal{T}\psi$ in $\mathcal{C}_b(H_+)$. ■

Existence and uniqueness for strong and strict solutions are stated below.

Corollary 5.2.14 *Consider the Dirichlet problem (5.1.1), then it holds for $\lambda > 0$:*
- *(i) for any datum $f \in \mathcal{C}_b(H_+)$, there exists a unique strong solution ψ and further $\psi = R(\lambda,\mathcal{T})f$;*
- *(ii) for any $g \in \mathcal{C}_s^2(H_+)$, we have that $R(\lambda,\mathcal{T})g$ is a strict solution.*

Proof (i) Let $f \in \mathcal{C}_b(H_+)$, we verify that $\psi = R(\lambda, \mathcal{T})f$ is a strong solution. By formula (5.2.35), we know that it is possible to choose $(\psi_n) \subset D(\mathcal{T}_0)$ such that
$\lim_{n\to\infty} \psi_n = \psi$ and $\lim_{n\to\infty} (\lambda - \mathcal{T}_0)\psi_n = (\lambda - \mathcal{T})\psi = f$ in $\mathcal{C}_b(H_+)$.
Now we prove the uniqueness. Let ϕ be a strong solution of (5.1.1) associated to the datum f. Then there exists $(\phi_n) \subset D(\mathcal{T}_0)$ such that

$$\lim_{n\to\infty} \phi_n = \phi, \quad \lim_{n\to\infty} \mathcal{T}_0\phi_n = \lambda\phi - f \text{ in } \mathcal{C}_b(H_+).$$

Since $\mathcal{T}$ is a closed operator, we find that $\phi \in D(\mathcal{T})$ and $\mathcal{T}\phi = \lambda\phi - f$. It follows that $\phi = R(\lambda, \mathcal{T})f$.
(ii) Let $g \in \mathcal{C}_s^2(H_+)$, by Theorem 5.2.11, we know that $\eta = R(\lambda, \mathcal{T})g \in D(\mathcal{T}_0)$. Using Theorem 5.2.13, we get $g = (\lambda - \mathcal{T})\eta = (\lambda - \mathcal{T}_0)\eta$ and so η is a strict solution.
∎

In the sequel we will study regularity properties of strong solutions.

5.3 Optimal regularity for the Dirichlet problem

5.3.1 Preliminary Schauder estimates

In this section we give Schauder estimates for $\psi = R(\lambda, \mathcal{T})f$, $f \in \mathcal{C}_b(H_+)$, $\lambda > 0$, when f vanishes on ∂H_+, see Theorem 5.3.5.

In this case we can deduce optimal regularity for (5.1.1) as a consequence of Schauder estimates given in Priola and Zambotti [70], see Chapter 4, on the whole space H. This approach is based on the fact that any $f \in \mathcal{C}_0(H_+)$ can be extended, by the operator E, to an odd function on H, with respect ot x_1, that belongs to $\mathcal{C}_b(H)$.

We need some preliminaries facts that can be easily verified. First we introduce odd and even functions on a Hilbert space, with respect to x_1. We recall that $\phi_1((x_1, x')) = (-x_1, x')$, $\forall x = (x_1, x') \in H$.

$$AS(H) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_b(H) \ / \ f(\phi_1(x)) = -f(x), \quad \forall x \in H\};$$

$$S(H) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_b(H) \ / \ f(\phi_1(x)) = f(x), \quad \forall x \in H\}.$$

It is simple to verify that $AS(H)$ and $S(H)$ are closed in $\mathcal{C}_b(H)$. Morevorer $f \in \mathcal{C}_0(H_+)$ implies that $Ef \in AS(H)$, $f \in \mathcal{C}_b(H_+)$. We define the following linear operators in $\mathcal{C}_b(H)$,

$$P_{AS}\, g \stackrel{\text{def}}{=} \frac{g(x) - g(\phi_1(x))}{2}, \quad P_S\, g \stackrel{\text{def}}{=} \frac{g(x) + g(\phi_1(x))}{2}, \ g \in \mathcal{C}_b(H), \ x \in H. \tag{5.3.1}$$

It is easy to verify that P_S and P_{AS} are contractions and that the following statements hold:

$$\begin{array}{l} P_{AS} : \mathcal{C}_b(H) \to AS(H), \quad P_S : \mathcal{C}_b(H) \to S(H); \\ P_{AS}^2 = P_{AS}, \ \ P_S^2 = P_S, \quad P_S = I - P_{AS}, \quad \mathrm{Ker}(P_{AS}) = P_S(H) = S(H). \end{array} \tag{5.3.2}$$

Proposition 5.3.1 *Let $\mathcal{A}$ be the generator of O_t, then it holds:*
(i) $\mathcal{C}_b(H) = AS(H) \oplus S(H)$ *(topological sum);*
(ii) $O_t\,(AS(H)) \subset AS(H), \quad t \geq 0$;
(iii) $\mathcal{A}\,P_{AS} = P_{AS}\,\mathcal{A}$.
Properties (ii) and (iii) also hold with $AS(H)$ replaced by $S(H)$.

Proof (i) This assertion follows easily by (5.3.2),
(ii) From Proposition 5.2.1 we know that:

$$O_t(f \circ \phi_1)(x) = O_t f(\phi_1(x)), \quad \forall f \in \mathcal{C}_b(H), \quad x \in H.$$

This yields $O_t g(\phi_1(x)) = O_t(g \circ \phi_1)(x) = -O_t g(x)$, for $x \in H$ and $g \in AS(H)$. It follows that $O_t g \in AS(H)$ for any $g \in AS(H)$, $t \geq 0$.
(iii) In a similar way to (ii), we can deduce that $P_{AS}O_t = O_t P_{AS}$, $t \geq 0$. Using this fact, we can obtain

$$P_{AS}\mathcal{A}f = P_{AS}\left(\lim_{t \to 0^+} \frac{O_t f - f}{t}\right) = \mathcal{A}P_{AS}f, \quad f \in D(\mathcal{A}). \ \blacksquare$$

Definition 5.3.2 The previous result allows to state that the restriction of O_t to $AS(H)$ (or to $S(H)$) is again a strongly continuous semigroup on $AS(H)$ (or $S(H)$).

We denote by $\hat{O}_t$ the restriction of O_t to $AS(H)$ and by $\hat{\mathcal{A}}$, its infinitesimal generator.
It is clear that: $D(\hat{\mathcal{A}}) = D(\mathcal{A}) \cap AS(H)$ and $\hat{\mathcal{A}}f = \mathcal{A}f$, for $f \in D(\hat{\mathcal{A}})$.

At the end of Proposition 5.2.6 we have pointed out that $\mathcal{C}_0(H_+)$ is the maximal subspace on which P_t is a strongly continuous semigroup. Now we denote by $\hat{P}_t$, the restriction of P_t to $\mathcal{C}_0(H_+)$ and by $\hat{\mathcal{T}}$ its infinitesimal generator. One can shown that $\hat{P}_t$ is the *part* of P_t in $\mathcal{C}_0(H_+)$, this means that

$$D(\hat{\mathcal{T}}) = \{g \in D(\mathcal{T}) \text{ such that } \mathcal{T}g \in \mathcal{C}_0(H_+)\}. \ \blacksquare$$

We point out that $\hat{P}_t$ and $\hat{O}_t$ are *isomorphic semigroups.*
Indeed there exists an isomorphism (that is also an isometry),
$E : \mathcal{C}_0(H_+) \to AS(H)$ with $E^{-1} = R$ and $\hat{P}_t = R\hat{O}_t E$.
Thus we have $D(\hat{\mathcal{A}}) = E\,[D(\hat{\mathcal{T}})]$, $\hat{\mathcal{A}} = E\,\hat{\mathcal{T}}\,R$.

Let L be a closed operator on a Banach space X, we recall that $\rho(L)$ denotes the resolvent set of L, see also (4.2.1). Moreover we denote by $L|_G$, the restriction of L to a subspace G of X. We need the following straightforward result.

Lemma 5.3.3 *Following the notations of Definition 5.3.2, one has*
(i) $\rho(\mathcal{A}) \subset \rho(\hat{\mathcal{A}})$;
(ii) $R(\lambda, \hat{\mathcal{A}}) = R(\lambda, \mathcal{A})\,|_{AS(H)}$, *for* $\lambda \in \rho(\mathcal{A})$.

Proof Let $\lambda \in \rho(\mathcal{A})$, then $(\lambda - \mathcal{A}) : D(\mathcal{A}) \to \mathcal{C}_b(H)$ is an isomorphism and so in particular $(\lambda - \hat{\mathcal{A}})$, the restriction of $(\lambda - \mathcal{A})$ to $D(\hat{\mathcal{A}}) = D(\mathcal{A}) \cap AS(H)$, is one to one.

We have $(\lambda - \hat{\mathcal{A}}) : D(\hat{\mathcal{A}}) \to AS(H)$. We verify that $(\lambda - \hat{\mathcal{A}})$ is onto. This will imply, by the Open Mapping Theorem, that $(\lambda - \hat{\mathcal{A}})$ is an isomorphism and that $R(\lambda, \hat{\mathcal{A}}) = R(\lambda, \mathcal{A})|_{AS(H)}$.

Take any $g \in AS(H)$ and set $f = R(\lambda - \mathcal{A})g \in D(\mathcal{A})$. Using (iii) of Proposition 5.3.2, we deduce

$$g = P_{AS}\, g = P_{AS}\, (\lambda - \mathcal{A})f = (\lambda - \mathcal{A})P_{AS}\, f.$$

Therefore $P_{AS}\, f = f$ and $(\lambda - \hat{\mathcal{A}})f = g$. The proof is complete. ∎

We prepare the main result of this section by a preliminary lemma concerning $Q-$Hölder functions.

Lemma 5.3.4 *For any $f \in \mathcal{C}^{\theta}_Q(H_+) \cap \mathcal{C}_0(H_+)$, $\theta \in (0,1)$ we have $Ef \in \mathcal{C}^{\theta}_Q(H) \cap AS(H)$*
and $\|Ef\|_{\theta,Q} \le 2\, \|f\|_{\theta,Q}$.

Proof Fix $f \in \mathcal{C}^{\theta}_Q(H_+) \cap \mathcal{C}_0(H_+)$, then it is clear that $Ef \in \mathcal{C}_b(H)$ and moreover $\|Ef\|_0 = \|f\|_0$.

We verify the $Q-$Hölder condition. For any u, $u + Q^{1/2}v \in \overline{H_+}$, it holds

$$|Ef(u) - Ef(u + Q^{1/2}v)| = |f(u) - f(u + Q^{1/2}v)| \le \|f\|_{\theta,Q}|v|^{\theta};$$

For any $u, u + Q^{1/2}v \in H_-$, one has

$$\begin{aligned} &|Ef(u) - Ef(u + Q^{1/2}v)| = | - f(\phi_1(u)) + f(\phi_1(u) + Q^{1/2}\phi_1(v))| \\ &\le \|f\|_{\theta,Q}\, |\phi_1(v)|^{\theta} = \|f\|_{\theta,Q}|v|^{\theta}. \end{aligned}$$

Finally suppose that $u \in H_-$ and $u + Q^{1/2}v \in \overline{H_+}$ and define $[u, u + Q^{1/2}v] = \{x \in H \;/\; x = u + rQ^{1/2}v$ with $r \in [0,1]\ \}$. There exists a unique $z \in [u, u + Q^{1/2}v] \cap \partial H_+$, $z = u + \hat{r}Q^{1/2}v$ with $\hat{r} \in [0,1]$. Hence we obtain:

$$\begin{aligned} &|Ef(u) - Ef(u+Q^{1/2}v)| \le |f(u) - f(z)| + |Ef(u+Q^{1/2}v) - Ef(z)| \le \|f\|_{\theta,Q}[\,|\hat{r}v|^{\theta} \\ &+ |(1-\hat{r})v|^{\theta}\,] \le 2\|f\|_{\theta,Q}\, |v|^{\theta}. \end{aligned}$$ ∎

We point out that if $f \in \mathcal{C}^1_Q(H_+) \cap \mathcal{C}_0(H_+)$, we have that $Ef \in \mathcal{C}^1_Q(H)$. However $f \in \mathcal{C}^2_Q(H_+) \cap \mathcal{C}_0(H_+)$ does not imply that $Ef \in \mathcal{C}^2_Q(H)$.

Now we are ready to prove a first version of Schauder estimates, that improves Theorem 4.2 in Priola [66].

Theorem 5.3.5 *Consider $\psi = R(\lambda, \mathcal{T})f$, $f \in \mathcal{C}^{\theta}_Q(H_+) \cap \mathcal{C}_0(H_+)$, $\lambda > 0$, $\theta \in (0,1)$.*

Then $\psi \in \mathcal{C}^2_Q(H_+) \cap \mathcal{C}_0(H_+)$, $D^2_Q\psi \in \mathcal{C}^\theta_Q(H_+, \mathcal{L}_2(H))$ *and there exists a constant* $c = c(\theta, Q, \lambda) > 0$ *such that*

$$\|\psi\|_{2,Q} + \|D^2_Q\psi\|_{\theta,Q,\mathcal{L}_2(H)} + \lambda\|\psi\|_{\theta,Q} + \|\mathcal{T}\psi\|_{\theta,Q} \leq c\,\|f\|_{\theta,Q}. \tag{5.3.3}$$

Moreover $\hat{D}^2_Q\psi(0, x') = D_{11}\psi(0, x') = 0$, *for any* $x' \in H'$.

Proof By Lemma 5.3.4, we know that $Ef \in \mathcal{C}^\theta_Q(H) \cap AS(H)$. First we verify that

$$\lambda\|\psi\|_{\theta,Q} + \|\mathcal{T}\psi\|_{\theta,Q} \leq c\|f\|_{\theta,Q}.$$

This fact follows, since

$$\|P_t f\|_{\theta,Q} \leq \|O_t Ef\|_{\theta,Q} \leq \|Ef\|_{\theta,Q} \leq 2\|f\|_{\theta,Q}, \quad t \geq 0$$

and consequently

$$\|\psi\|_{\theta,Q} \leq \int_0^\infty e^{-\lambda s}\,\|f\|_{\theta,Q}\,ds = \frac{2}{\lambda}\|f\|_{\theta,Q}.$$

In order to prove the estimate (5.3.3), we will apply Proposition 4.2.2. Since $Ef \in \mathcal{C}^\theta_Q(H) \cap AS(H)$, we know that

$$\phi \stackrel{\text{def}}{=} (\lambda - \mathcal{A})^{-1}\,Ef \in \mathcal{C}^2_Q(H) \text{ and } D^2_Q\phi \in \mathcal{C}^\theta_Q(H, \mathcal{L}_2(H)).$$

Moreover there exists $c = c(\theta, Q, \lambda) > 0$ such that

$$\|\phi\|_{2,Q} + \|D^2_Q\phi\|_{\theta,Q,\mathcal{L}_2} \leq c\,\|Ef\|_{\theta,Q} \leq 2c\|f\|_{\theta,Q}.$$

Now, using the isomorphic semigroups $\hat{O}_t$ and $\hat{P}_t$, see Definition 5.3.2, and Lemma 5.3.3, we find

$$\begin{aligned}\phi &= (\lambda - \mathcal{A})^{-1}\,Ef = (\lambda - \hat{\mathcal{A}})^{-1}\,Ef = (\lambda - E\hat{\mathcal{T}}R)^{-1}\,Ef\\ &= E(\lambda - \hat{\mathcal{T}})^{-1}\,REf = E(\lambda - \hat{\mathcal{T}})^{-1}\,f = E\psi.\end{aligned}$$

It follows that $\psi \in \mathcal{C}^2_Q(H_+)$ and $D^2_Q\psi \in \mathcal{C}^\theta_Q(H_+, \mathcal{L}_2(H))$. Morever one has

$$\|\psi\|_{2,Q} + \|D^2_Q\psi\|_{\theta,Q,\mathcal{L}_2} \leq \|E\psi\|_{2,Q} + \|D^2_Q E\psi\|_{\theta,Q,\mathcal{L}_2} \leq 2\,c\|f\|_{\theta,Q}.$$

Finally in order to prove that $D_{11}\psi(0, x') = 0$, we remark that it holds

$$D_{11}\psi(0, x') = \int_0^{+\infty} e^{-\lambda t}\; D_{11}O_t Ef(0, x')\,dt, \quad x' \in H'. \tag{5.3.4}$$

Indeed, by formula 4.2.8, we know that

$$\begin{aligned}\|D_{11}O_t Ef\|_0 \leq \|D^2_Q O_t Ef\|_{0,\mathcal{L}_2} &\leq c_\theta\, t^{\theta/2-1}\,\|Ef\|_{\theta,Q}\\ &\leq 2c_\theta\, t^{\theta/2-1}\,\|f\|_{\theta,Q}, \quad t > 0;\end{aligned}$$

thanks to this estimate we can differentiate under the integral sign and obtain (5.3.4). Now we can verify directly that for any $x' \in H'$, $t > 0$, it holds

$$D_{11}\Big(\int_{\mathbb{R}_+} \frac{(e^{-\frac{(x_1-y_1)^2}{2t\lambda_1}} - e^{-\frac{(x_1+y_1)^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}}\; dy_1\Big)(0,x') \;=0.$$

This yields, taking the integral representation of O_tEf and differentiating under the integral sign,

$$D_{11}O_tEf(0,x') \;=\; 0, \quad f \in \mathcal{B}_b(\overline{H_+}), \quad x' \in H', \;\; t>0.$$

Using this fact in (5.3.4), we find that $D_{11}\psi(0,x') \;\;= 0, \;\; x' \in H'$. The proof is complete. ∎

5.3.2 Schauder estimates: general case

Consider the equation

$$(\lambda - \mathcal{T})\,\psi \;=\; f, \quad \text{where} \quad f \in \mathcal{C}^\theta_Q(H_+), \;\; \theta \in (0,1).$$

We are not able to prove that the solution $\psi \in \mathcal{C}^2_Q(H_+)$ and $D^2_Q\psi \in \mathcal{C}^\theta_Q(H_+, \mathcal{L}_2(H))$ as in the previous section. We need an additional assumption, namely that the restriction of f to ∂H_+ is θ- Hölder continuous. Under this assumption we can obtain optimal regularity for ψ. Thus we define the following new space.

Definition 5.3.6 Let $(E, \|\cdot\|_E)$ be a Banach space. We define $\mathcal{N}^\theta_Q(H_+, E)$, with $\theta \in (0,1)$, as the set of all functions $f \in \mathcal{C}^\theta_Q(H_+, E)$ such that $f(0,\cdot) \in \mathcal{C}^\theta_b(H', E)$. $\mathcal{N}^\theta_Q(H_+, E)$ is a Banach space equipped with the norm:

$$\|f\|_{\theta,\mathcal{N},E} \;=\; [f(0,\cdot)]_{\theta,E} \;+\; \|f\|_{\theta,Q,E}, \quad f \in \mathcal{N}^\theta_Q(H_+, E),$$

where $[f(0,\cdot)]_{\theta,E}$ is the Hölder constant of $f(0,\cdot)$. When $E = \mathbb{R}$, we set $\mathcal{N}^\theta_Q(H_+, \mathbb{R}) = \mathcal{N}^\theta_Q(H_+)$. ▪

Identifying $\mathcal{C}_b(H')$ with the subspace of $\mathcal{C}_b(H_+)$ of all functions which are constant in the first variable (see formula (5.1.3)), we can split the Dirichlet problem

$$\lambda\varphi(x) \;-\; \mathcal{T}\varphi(x) = f(x), \quad f \in \mathcal{C}_b(H_+), \;\; x \in H_+, \; \lambda > 0 \tag{5.3.5}$$

into two problems:

$$\lambda\phi(x) \;-\; \mathcal{T}\phi(x) = g(x), \quad g \in \mathcal{C}_0(H_+), \;\; x \in H_+, \tag{5.3.6}$$

$$\lambda\psi(x) \;-\; \mathcal{T}\psi(x) = h(x), \quad h \in \mathcal{C}_b(H'), \;\; x \in H_+, \tag{5.3.7}$$

where $g(x) \;=\; f(x) - f(0,x')$ and $h(x') \;=\; f(0,x')$. Thus it is clear that solving problems (5.3.6) and (5.3.7) is the same as solving (5.3.5). The problem (5.3.6) has been studied in the previous section; now we are concerned with regularity properties for the strong solution ψ of (5.3.7).

We shall use the notation of part (b) of Theorem 5.2.11: $P_t h(x) \ = \ \eta_t(x_1)\, O'_t h(x'),\ x \in H_+,\ h \in \mathcal{C}_b(H'),$

$$\text{where} \quad \eta_t(x_1) \stackrel{\text{def}}{=} U_t 1(x_1) \ = \ \int_0^{x_1} \frac{2e^{-\frac{u^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}} du,\ x_1 \geq 0,\ t > 0.$$

Let $g \in \mathcal{C}^1_Q(H_+)$, we set $\ < D_Q g(x), v >=< D_{Q'} g(x), v' > + D_1 g(x)\, v_1,\ \ x \in H_+,$
where $\ < D_{Q'} g(x), v' > \stackrel{\text{def}}{=} \sum_{h=2}^{\infty} \sqrt{\lambda_h}\ D_h g(x) v'_h,\ \ x \in H_+,\ \ v = (v_1, v') \in H.$

Let $f \in \mathcal{C}^2_Q(H_+)$, we consider three different symmetric operators in $\mathcal{L}(H)$:

$$< D^2_{Q'} f(x)(u), v > \stackrel{\text{def}}{=} \sum_{h,k=2}^{\infty} \sqrt{\lambda_k}\sqrt{\lambda_h}\ D_{kh} f(x) u_k v_h,\ \ x \in H_+,\ u, v \in H,$$

$$\begin{aligned} &< D_1 D_{Q'} f(x)(u), v > \stackrel{\text{def}}{=} \sum_{h=2}^{\infty} \sqrt{\lambda_1}\sqrt{\lambda_h} D_{1h} f(x)[u_1 v_h + v_1 u_h], \quad x \in H_+,\ \ u, v \in H, \\ &< D_{11} f(x)(u), v > \ \stackrel{\text{def}}{=} D_{11} f(x) u_1 v_1, \quad x \in H_+,\ \ u, v \in H. \end{aligned} \tag{5.3.8}$$

Clearly $\ D^2_Q f(x) \ = \ D^2_{Q'} f(x) \ + \ D_1 D_{Q'} f(x) + D_{11} f(x), \quad x \in H_+.$
Moreover we define:

$$\tilde{D}^2_Q f(x) \stackrel{\text{def}}{=} D^2_{Q'} f(x) \ + \ D_1 D_{Q'} f(x), \quad f \in \mathcal{C}^2_Q(H_+), \quad x \in H_+. \tag{5.3.9}$$

Definition 5.3.7 We define an operator $S : \mathcal{B}_b(\overline{H_+}) \to \mathcal{B}_b(H)$; for any $\ f \in \mathcal{B}_b(\overline{H_+}),$

$$S f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & if \ \ x \in \overline{H}_+ \\ \\ f(\phi_1(x)) & if \ \ x \in H_-. \end{cases} \tag{5.3.10}$$

Then we introduce the semigroup $\ T_t$ on $\mathcal{C}_b(H_+)$: $\qquad T_t \stackrel{\text{def}}{=} R O_t S.$
Since for any $f \in \mathcal{C}_b(H_+)$, we have that $Sf \in \mathcal{C}_b(H)$, it turns out that T_t is a strongly continuous semigroup on $\mathcal{C}_b(H_+)$. Moreover T_t admits the following integral representation

$$T_t f(x) = \int_{\mathbb{R}_+} \left(\frac{(e^{-\frac{(x_1-y_1)^2}{2t\lambda_1}} + e^{-\frac{(x_1+y_1)^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}} \right) dy_1 \int_{H'} f(y_1, x' + y') \mathcal{N}(0', tQ') dy',\ x \in H_+, \tag{5.3.11}$$

where $f \in \mathcal{C}_b(H_+)$. In the sequel we only need a connection between P_t and T_t. ∎

The next lemma will be used to prove the uniform continuity for Laplace transforms, involving Hilbert-Schmidt operators.

Lemma 5.3.8 *Fix any $\lambda > 0$ and consider the Lebesgue measure dt on $\mathbb{R}$. Let $W : [0, \infty) \times H_+ \to \mathcal{L}_2(H)$, such that:*

(i) $\sup_{x \in H_+} \|W(t, x)\|_{\mathcal{L}_2(H)} \le f(t)$, *where f is a real integrable map on $[0, \infty)$ with respect to $e^{-\lambda t} dt$;*
(ii) for any $x \in H_+$, $u, v \in H$, the map $< W(\cdot, x)(u), v >: [0, \infty) \to \mathbb{R}$, $t \mapsto < W(t, x)(u), v >$ is Borel;
(iii) for fixed $t > 0$, the map: $W(t, \cdot) : H_+ \to \mathcal{L}_2(H)$, $x \mapsto W(t, x)$ belongs to $\mathcal{C}_b(H_+, \mathcal{L}_2(H))$.

Then the map $T : H_+ \to \mathcal{L}(H)$ defined by

$$< T(x)(u), v > = \int_0^\infty e^{-\lambda t} < W(t, x)(u), v > dt, \quad x \in H_+, \quad u, v \in H,$$

belongs to $\mathcal{C}_b(H_+, \mathcal{L}_2(H))$.

Proof The proof is similar to that of Proposition 4.2.1. Denote by $\mathcal{F}_1$ the subspace of $\mathcal{L}(H)$ of all finite rank operators N, such that $\|N\|_{\mathcal{L}_2(H)} \le 1$. We show that $T(x)$ is of Hilbert-Schmidt type by using (4.2.4), see also Lemma 1.1.3. To this purpose take $N \in \mathcal{F}_1$ and choose an orthonormal basis $(l_k), k = 1, ..n$ in $N(H)$. There results, for any $x \in H_+$,

$$\begin{aligned} |\text{Tr } (NT(x))| &= \Big| \sum_{k=1}^n < T(x)(l_k), N^* l_k > \Big| \\ &\le \int_0^\infty e^{-\lambda t} \, |\text{Tr } (NW(t, x))| \, dt \le \int_0^\infty e^{-\lambda u} f(u) \, du \end{aligned} \tag{5.3.12}$$

Taking the supremum over all $N \in \mathcal{F}_1$, we find that $T(x) \in \mathcal{L}_2(H)$, $x \in H_+$, and moreover $\sup_{x \in H_+} \|T(x)\|_{\mathcal{L}_2(H)} < \infty$.

It remains to establish the uniform continuity of T. This is equivalent to show that for any sequence $(z_n) \subset \overline{H}_+$ such that $z_n \to 0$ as $n \to \infty$, it holds:

$$\lim_{n \to \infty} \sup_{x \in H_+} \|T(x + z_n) - T(x)\|_{\mathcal{L}_2(H)} = 0. \tag{5.3.13}$$

Let us fix a countable dense subset L of H_+. Since $\mathcal{L}_2(H)$ is separable, we also choose a countable dense subset $\mathcal{M}$ of $\mathcal{F}_1$. Now we have

$$\begin{aligned} \gamma_n(t) &\stackrel{\text{def}}{=} \sup_{x \in H_+} \|W(t, x + z_n) - W(t, x)\|_{\mathcal{L}_2(H)} \\ &= \sup_{x \in L, \, N \in \mathcal{M}} |\text{Tr } (N[W(t, x + z_n) - W(t, x)])|, \quad t > 0. \end{aligned}$$

Indeed $W(t, \cdot)$ is uniformly continuous in H_+ and further the linear map : $\mathcal{L}_2(H) \to \mathbb{R}$, $N \mapsto \text{Tr } (NA)$ is continuous, $A \in \mathcal{L}_2(H)$. The maps γ_n are Borel, since $L \times \mathcal{M}$ is countable. Thus we can write

$$\sup_{x \in H_+} \|T(x + z_n) - T(x)\|_{\mathcal{L}_2(H)} \le \int_0^\infty e^{-\lambda t} \, \gamma_n(t) \, dt, \quad n \ge 1.$$

Now $\lim_{n\to\infty} \gamma_n(t) = 0$, $t > 0$, by (iii), and moreover $\gamma_n(t) \le 2f(t)$, $t \ge 0$, $n \ge 1$. Letting $n \to \infty$ in the right-hand side of the last formula, we find (5.3.13) by the Dominated Convergence Theorem. This completes the proof. ∎

Now we consider the real interpolation spaces, introduced in Section 1.4. In Cannarsa and Da Prato [12, §5.1], it is proved that

$$\mathcal{C}_Q^\theta(H) \;=\; (\mathcal{C}_b(H), \mathcal{C}_Q^1(H))_{\theta,\infty}, \quad \theta \in (0,1). \tag{5.3.14}$$

We also use the spaces $\mathcal{C}_{Q'}^1(H')$ and $\mathcal{C}_{Q'}^\theta(H')$, $\theta \in (0,1)$, that can be clearly defined as in Section 1.3, replacing H and Q by H' and Q'. However in the sequel, similarly to $\mathcal{C}_b(H')$ (see formula 5.1.3), we will identify $\mathcal{C}_{Q'}^1(H')$ and $\mathcal{C}_{Q'}^\theta(H')$ respectively with the closed subspaces of $\mathcal{C}_Q^1(H_+)$ and $\mathcal{C}_Q^\theta(H_+)$ of all functions constant in the first variable. We recall that from Proposition 3.2.3, there results for $t > 0$:

$$\begin{aligned}
&\|D_Q^2 O_t h\|_{0,\mathcal{L}_2(H)} \le \frac{2}{t}\|h\|_0, \;\; \|D_Q^2 O_t g\|_{0,\mathcal{L}_2(H)} \le \frac{1}{\sqrt{t}}\|g\|_{1,Q}, \;\; h \in \mathcal{C}_b(H),\, g \in \mathcal{C}_Q^1(H),\\
&\|D_Q O_t f\|_0 \;\le\; \frac{1}{\sqrt{t}}\|f\|_0, \quad f \in \mathcal{C}_b(H).
\end{aligned} \tag{5.3.15}$$

By Theorem 1.4.1, interpolating between the first two estimates, thanks to (5.3.14), one has

$$\|D_Q^2 O_t f\|_{0,\mathcal{L}_2(H)} \;\le\; c_\theta\, t^{\theta/2-1}\, \|f\|_{\theta,Q}, \;\; t > 0. \tag{5.3.16}$$

Now we prove a non optimal version of Schauder estimates.

Theorem 5.3.9 *Consider* $\psi = R(\lambda, \mathcal{T})h$, $h \in \mathcal{C}_{Q'}^\theta(H')$ $\lambda > 0$, $\theta \in (0,1)$. *Then* $\psi \in \mathcal{C}_Q^2(H_+)$ *and there exists a constant* $c = c(\lambda, Q, \theta) > 0$*, such that:*

$$\|\psi\|_{2,Q} \;+ \|D_Q^2\psi\|_{0,\mathcal{L}_2} \;\le\; c\;\|h\|_{\theta,Q}.$$

Proof Let $h \in \mathcal{C}_{Q'}^\theta(H')$, we have

$$\psi(x) \;=\; \int_0^\infty e^{-\lambda t}\; \eta_t O_t' h\,(x)\; dt, \;\; x \in H_+. \tag{5.3.17}$$

Arguing as in part (b) of the proof of Theorem 5.2.11, first we estimate $D_Q P_t h$,

$$\|D_Q(\eta_t O_t' h)\|_0 \;\le\; c_1\, \frac{1}{\sqrt{t}}\, \|h\|_0, \quad t > 0.$$

Thus, differentiating under the integral sign, we easily deduce that

$$\psi \in \mathcal{C}_Q^1(H) \quad \text{and} \quad \|D_Q\psi\|_0 \;\le\; c_2\|h\|_0.$$

Then for any $u, v \in H$ we have, recalling (5.3.8),

$$\begin{aligned}
&< D_Q^2(\eta_t O_t' h)(x)\;(v), u > \;=\; \lambda_1 D_{11}\eta_t(x_1) v_1 u_1\; O_t' h(x')\\
&\;+\; < D_{Q'}^2 P_t h(x)(u), v > \;+\; < D_1 D_{Q'} P_t h(x)(u), v >
\end{aligned}$$

where

$$< D^2_{Q'} P_t h(x)(u), v > = \ \eta_t(x_1) < D^2_{Q'} O'_t h(x')(v'), u' > \quad \text{and}$$

$$< D_1 D_{Q'} P_t h(x)(u), v > = \ \sqrt{\lambda_1}\, [D_1\eta_t(x_1) v_1 < D_{Q'} O'_t h(x'), u' >$$
$$+ D_1\eta_t(x_1) u_1 < D_{Q'} O'_t h(x'), v' >], \quad x \in H_+, \quad u, v \in H.$$

Now using (5.3.14), with H, Q replaced by H', Q' and proceeding as in (5.3.16), we deduce that

$$\|D^2_{Q'} P_t h\|_{0,\mathcal{L}_2} = \|D^2_{Q'}(\eta_t O'_t h)\|_{0,\mathcal{L}_2} \le \|D^2_{Q'} O'_t h\|_{0,\mathcal{L}_2} \le \ c'_\theta t^{\theta/2-1}\ \|h\|_{\theta,Q}, \ \ t > 0. \tag{5.3.18}$$

To estimate $D_1 D_{Q'} P_t h$, let us notice that, according to formula (5.3.15),

$$\begin{aligned} \|D_1 D_{Q'} P_t f\|_{0,\mathcal{L}_2} &\le 2\|D_1\eta_t\|_0 \|D_{Q'} O'_t f\|_0 \le \frac{c}{t}\|f\|_0, \ \ f \in \mathcal{C}_b(H'), \ t > 0, \\ \|D_1 D_{Q'} P_t g\|_{0,\mathcal{L}_2} &\le \ 2\|D_1\eta_t\|_0 \|D_{Q'} O'_t g\|_0 \ \le \ \frac{c}{\sqrt{t}}\|D_Q g\|_0, \ \ g \in \mathcal{C}^1_{Q'}(H'), \ t > 0, \end{aligned}$$

where $c = c(Q)$. Interpolating between these two estimates, by Theorem 1.4.1, it follows that there exists a constant $c_4 = c_4(\theta, Q) > 0$ such that

$$\|D_1 D_{Q'} P_t h\|_{0,\mathcal{L}_2} \ \le \ c_4\ t^{\theta/2-1}\|h\|_{\theta,Q}, \ \ t > 0. \tag{5.3.19}$$

Recalling that $H^\delta_+ \ = \ \{(x_1, x') \in H_+ \ \text{ such that } \ x_1 > \delta\}$ and also using formula (5.2.28) concerning $D_{11} P_t h$, we obtain

$$\sup_{x \in H^\delta_+} \ \|D^2_Q P_t h(x)\|_{\mathcal{L}_2(H)} \ \le \ c_5 \frac{t^{\theta/2-1}}{\delta}\|h\|_{\theta,Q}, \ \ t > 0, \ \delta > 0.$$

This estimate allows to differentiate in (5.3.17) under the integral sign and to obtain that there exists $D^2_Q\psi(x)$, for any $x \in H_+$,

$$< D^2_Q\psi(x)(u), v > = \int_0^\infty e^{-\lambda t} \ \ < D^2_Q P_t h(x)(u), v > \ dt, \quad x \in H_+, \quad u, v \in H.$$

Moreover, arguing as in Lemma 5.3.8, we get easily that $D^2_Q\psi(x) \in \mathcal{L}_2(H)$, $x \in H_+$. To get the uniform continuity of $D^2_Q\psi$, we split it, according to formula (5.3.9),

$$< D^2_Q\psi(x)(u), v > = < \tilde{D}^2_Q\psi(x)(u), v > \ + \lambda_1 D_{11}\psi(x) u_1 v_1, \ \ x \in H_+, \ u, v \in H.$$

Notice that:

$$< \tilde{D}^2_Q\psi(x)(u), v > = \int_0^\infty e^{-\lambda t} \ \ < \tilde{D}^2_Q P_t h(x)(u), v > \ dt, \quad x \in H_+, \quad u, v \in H,$$

$$D_{11}\psi(x) \;=\; -\frac{2}{\sqrt{2\pi\lambda_1}}\int_0^\infty \frac{x_1}{\lambda_1 t\sqrt{t}} e^{\frac{-x_1^2}{2t\lambda_1}}\, e^{-\lambda t}\, O_t' h(x')\, dt, \quad x \in H_+.$$

By Corollary 5.2.12 we know that $D_{11}\psi \in \mathcal{C}_b(H_+)$. Moreover, applying Lemma 5.3.8 thanks to estimates (5.3.18) and (5.3.19), we obtain that $D_Q^2\psi \in \mathcal{C}_b(H_+, \mathcal{L}_2(H))$. The proof is complete. ∎

Now we split again $D_Q^2\psi$ into $\tilde{D}_Q^2\psi$ and $D_{11}\psi$ and consider each of them separately. We need the following connection between T_t and P_t.

Lemma 5.3.10 *Let $f \in \mathcal{C}_0(H_+)\cap \mathcal{C}_Q^2(H_+)$ then for any $x \in H_+,\ u, v \in H$ it holds:*

(i) $< D_1 D_{Q'} P_t f(x)(u), v >= T_t(< D_1 D_{Q'} f(\cdot)(u), v >)(x), \quad t \geq 0;$
(ii) $< D_1 D_{Q'} T_t f(x)(u), v >= P_t(< D_1 D_{Q'} f(\cdot)(u), v >)(x), \quad t \geq 0.$

Proof By the definition of $D_1 D_{Q'}$, it is enough to verify that

$$D_{1h} P_t f(x) \;=\; T_t D_{1h} f(x), \quad f \in \mathcal{C}_0(H_+)\cap\mathcal{C}_Q^2(H_+),\ x \in H_+, \quad h \geq 2,\ t > 0.$$

To this purpose we compute for any $h \geq 2, \ \ x \in H_+, \ \ t > 0$,

$$\begin{aligned} D_{1h}P_t f(x) \;&= D_1\Big(\int_{H'}\Big[\int_{-x_1}^\infty \frac{e^{\frac{-(y_1)^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}} D_h f(x_1+y_1, x'+y')dy_1 \\ &- \int_{x_1}^\infty \frac{e^{-\frac{(y_1)^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}} D_h f(-x_1+y_1, x'+y')\, dy_1\Big]\mathcal{N}(0', tQ')dy'\Big) \\ &= \int_{H'}\Big[\int_{-x_1}^\infty \frac{e^{-\frac{(y_1)^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}} D_{1h} f(x_1+y_1, x'+y')dy_1 \\ &+ \int_{x_1}^\infty \frac{e^{-\frac{(y_1)^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}} D_{1h} f(-x_1+y_1, x'+y')\, dy_1\Big]\mathcal{N}(0', tQ')dy' \;=\; T_t D_{1h} f(x). \end{aligned} \tag{5.3.20}$$

Now, by standard arguments, we can get (i) and (ii). ∎

In order to prove the next theorem, we need the following result, proved in Cannarsa and Da Prato [12, §5.1], see also Section 3.4,

$$\mathcal{D}_{\mathcal{A}}(\theta/2, \infty) \hookrightarrow \mathcal{C}_Q^\theta(H), \ \ \theta \in (0,1), \quad \text{with a continuous embedding.}$$

Proposition 5.3.11 *Consider* $\psi \;=\; R(\lambda, \mathcal{T})h, \quad h \in \mathcal{C}_{Q'}^\theta(H'), \ \ \lambda > 0,\ \theta \in (0,1)$. *Then* $\psi \in \mathcal{C}_Q^2(H_+), \ \ \tilde{D}_Q^2\psi \in \mathcal{C}_Q^\theta(H_+, \mathcal{L}_2(H))$ *and there exists a constant* $c = c(\theta, Q, \lambda) > 0$ *such that*

$$\|\tilde{D}_Q^2\psi\|_{\theta,Q,\mathcal{L}_2(H)} \;\leq\; c\, \|h\|_{\theta,Q}.$$

Proof We recall that $\tilde{D}_Q^2 g = D_{Q'}^2 g + D_1 D_{Q'} g$, for any $g \in \mathcal{C}_Q^2(H_+)$. Using the notations of formula (5.3.12) (see also Lemma 1.1.3), the assertion will follow by proving the next statements, for any $N \in \mathcal{F}_1$, $\theta \in (0,1)$,

(i) $E[\mathrm{Tr}\,(ND_{Q'}^2\psi)] \in \mathcal{D}_{\mathcal{A}}(\theta/2,\infty)$, $\|E[\mathrm{Tr}\,(ND_{Q'}^2\psi)]\|_{(\theta/2,\mathcal{A})}$
$\le C'\,\|h\|_{\theta,Q}$,

(ii) $S[\mathrm{Tr}\,(N\,D_1 D_{Q'}\psi)] \in \mathcal{D}_{\mathcal{A}}(\theta/2,\infty)$, $\|S[\mathrm{Tr}\,(N\,D_1 D_{Q'}\psi)]\|_{(\theta/2,\mathcal{A})}$
$\le\; C\,\|h\|_{\theta,Q}$,

where $h \in \mathcal{C}_{Q'}^{\theta}(H')$, S is defined in (5.3.7) and $C = C(\lambda,Q,\theta) > 0$, $C' = C'(\lambda,Q,\theta) > 0$. To establish (i) we fix $N \in \mathcal{F}_1$ and remark that $\mathrm{Tr}\,(ND_{Q'}^2\psi(0,x')) = 0$, $x \in H'$. Thus we have $E[\mathrm{Tr}\,(ND_{Q'}^2\psi)] \in AS(H)$.

Then for any function $f \in \mathcal{C}_Q^2(H_+)$, we set:

$$Uf(x) \;=\; \mathrm{Tr}\,(ND_{Q'}^2 f(x)), \qquad x \in H_+.$$

Now for $\xi \in [0,1]$, we find

$$I_\xi \;=\; \sup_{x\in H} |O_\xi EU\psi(x) - EU\psi(x)| \;=\; \sup_{x\in H_+} |P_\xi U\psi(x) - U\psi(x)|,$$

since we are considering functions in $AS(H)$ (see also Lemma 5.2.1). Now by the formula

$$P_t f(x) = \int_{\mathbb{R}_+} \left(\frac{(e^{-\frac{(x_1-y_1)^2}{2t\lambda_1}} - e^{-\frac{(x_1+y_1)^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}} \right) dy_1 \int_{H'} f(y_1, x'+y')\,\mathcal{N}(0', tQ')dy',$$

where $f \in \mathcal{C}_b(H_+)$ and $x \in H_+$, by a simple computation, as in (4.2.14), we obtain that

$$P_t Uf(x) \;=\; UP_t f(x), \quad f \in \mathcal{C}_Q^2(H_+),\;\; x \in H_+,\;\; t \ge 0.$$

This yields, using (5.3.18) and proceeding as in (4.2.15),

$$\begin{aligned} I_\xi \;&=\; \sup_{x\in H_+} \Big| \int_0^\infty e^{-\lambda t} \Big(UP_{t+\xi}h(x) - UP_t h(x) \Big)\, dt \Big| \;\le\; c\,\|h\|_{\theta,Q'} \\ &\cdot \Big[(e^{\lambda\xi}-1) \int_0^\infty e^{-\lambda t} t^{\theta/2-1} dt \;+\; e^{\lambda\xi} \int_0^\xi e^{-\lambda t} t^{\theta/2-1} dt \Big] \;\le\; C\|h\|_{\theta,Q'}\,\xi^{\theta/2}. \end{aligned} \tag{5.3.21}$$

and assertion (i) is proved.

As concerns (ii), fixing $N \in \mathcal{F}_1$, we point out that $S[\mathrm{Tr}\,(N\,D_1 D_{Q'}\psi)] \in S(H)$. Similarly to the case (i), we set for any function $f \in \mathcal{C}_Q^2(H_+)$,

$$Vf(x) \;=\; \mathrm{Tr}\,(N\,D_1 D_{Q'} f(x)), \qquad x \in H_+.$$

Now for $\xi \in [0,1]$, we find

$$J_\xi \;=\; \sup_{x\in H} |O_\xi SV\psi(x) - SV\psi(x)| \;=\; \sup_{x\in H_+} |T_\xi V\psi(x) - V\psi(x)|.$$

By Lemma 5.3.10, we know that

$$T_tVg(x) \;=\; VP_tg(x), \quad g\in \mathcal{C}^2_Q(H_+)\cap \mathcal{C}_0(H_+),\;\; x\in H_+,\;\; t\geq 0,$$

that allows us to obtain, using (5.3.19) and arguing as in (5.3.21),

$$J_\xi = \sup_{x\in H_+} \Big| \int_0^\infty e^{-\lambda t}\,\Big(VP_{t+\xi}h(x) - VP_th(x)\Big)dt\Big| \;\leq C'\,\|h\|_{\theta,Q'}\;\xi^{\theta/2}. \tag{5.3.22}$$

Thus also assertion (ii) is proved. The proof is complete. ∎

Now taking into account Theorem 5.2.12, Theorem 5.3.5 and Proposition 5.3.11, we can formulate the following result.

Theorem 5.3.12 *Consider* $\psi = R(\lambda,\mathcal{T})f,\;\; f\in \mathcal{C}^\theta_Q(H_+),\; \theta\in(0,1),\;\; \lambda>0.$ *Then* $\psi \in \mathcal{C}^2_Q(H_+)$, $\tilde D^2_Q\psi \in \mathcal{C}^\theta_Q(H_+,\mathcal{L}_2(H))$ *and there exists a constant* $c = c(\theta,Q,\lambda)>0$ *such that:*

$$\|\psi\|_{2,Q} + \|\tilde D^2_Q\psi\|_{\theta,Q,\mathcal{L}_2} + \lambda\|\psi\|_{\theta,Q} + \|\mathcal{T}\psi\|_{\theta,Q} \leq c\,\|f\|_{\theta,Q}.$$

Now we deal with the regularity of $D_{11}\psi$.

Proposition 5.3.13 *Consider* $\psi = R(\lambda,\mathcal{T})h,\;\; h\in\mathcal{C}^\theta_b(H')\;\; \lambda>0,\; \theta\in(0,1).$ *Then* $D_{11}\psi \in \mathcal{C}^\theta_b(H_+)$ *and there exists a constant* $c = c(\lambda,Q,\theta)>0$, *such that* $\|D_{11}\psi\|_\theta \leq c\,\|h\|_\theta$.

Proof By Corollary 5.2.12 we already know that there exists $D_{11}\psi\in\mathcal{C}_b(H_+)$ and a constant $C = C(\lambda,Q)>0$, such that: $\|D_{11}\psi\|_0 \leq c\|h\|_0$.

To prove the Hölder continuity, we come back to the proof of Theorem 5.2.11 and use the same notations. For any $x_1\in\mathbb{R}_+$, $x',\; z'\in H'$ we have, taking into account formula (5.2.30),

$$|D_{11}\psi(x_1,x') - D_{11}\psi(x_1,z')| \leq$$

$$\leq \frac{2}{\sqrt{2\pi\lambda_1}}\int_0^\infty \frac{x_1}{\lambda_1 t\sqrt{t}}e^{\frac{-x_1^2}{2t\lambda_1}}\;e^{-\lambda t}\;|O'_th(x') - O'_th(z')|\;dt. \leq c_1[h]_\theta\;|x'-z'|^\theta, \tag{5.3.23}$$

where c_1 is independent of x_1.
Let us consider the first variable. For any $x_1,\;\; s\in\mathbb{R}_+$ and $w'\in H'$ we find

$$|D_{11}\psi(x_1,w') - D_{11}\psi(x_1+s,w')|$$

$$\leq \frac{4}{\sqrt{2\pi\lambda_1}\lambda_1}\int_0^\infty e^{-\frac{u^2}{2\lambda_1}} \left| e^{-\lambda\,\frac{(x_1+s)^2}{u^2}}\, O'_{\frac{(x_1+s)^2}{u^2}} h(w') - \, e^{-\lambda\,\frac{(x_1)^2}{u^2}}\, O'_{\frac{x_1^2}{u^2}} h(w') \right| du$$

$$\leq c_{\lambda_1}\int_0^\infty e^{-\frac{u^2}{2\lambda_1}}\, e^{-\lambda\,\frac{(x_1+s)^2}{u^2}} \left| O'_{\frac{(x_1+s)^2}{u^2}} h(w') \; - \; O'_{\frac{(x_1)^2}{u^2}} h(w') \right| du \tag{5.3.24}$$

$$+ c_{\lambda_1}\int_0^\infty e^{-\frac{u^2}{2\lambda_1}} \left| e^{-\lambda\,\frac{(x_1+s)^2}{u^2}} \; - \; e^{-\lambda\,\frac{(x_1)^2}{u^2}} \right| O'_{\frac{x_1^2}{u^2}} h(w')\, du \; = \; \Gamma_1 \; + \; \Gamma_2.$$

To estimate Γ_2 we use the following explicit computation, see also Remark 5.3.14,

$$\frac{2}{\lambda_1} e^{-\sqrt{\frac{2\lambda}{\lambda_1}}x_1} \; = \; \frac{4}{\sqrt{2\pi\lambda_1}\lambda_1}\int_0^\infty e^{-\frac{u^2}{2\lambda_1}}\, e^{-\lambda\,\frac{x_1^2}{u^2}}\, du, \;\; x_1 \geq 0. \tag{5.3.25}$$

Thus we get for any $x_1 \in \mathbb{R}_+, \;\; s \in \mathbb{R}_+$

$$\Gamma_2 \; \leq c_{\lambda_1}\, \|h\|_0 \int_0^\infty e^{-\frac{u^2}{2\lambda_1}} \left(e^{-\lambda\,\frac{(x_1)^2}{u^2}} - e^{-\lambda\,\frac{(x_1+s)^2}{u^2}} \right) du \; \leq \; c_2\, \|h\|_0\, s, \tag{5.3.26}$$

where $c_2 = c_2(\lambda_1, \lambda)$. It remains to estimate Γ_1. We use the following inequality

$$\begin{aligned} &\left| O'_{\frac{(x_1+s)^2}{u^2}} h(w') \; - \; O'_{\frac{(x_1)^2}{u^2}} h(w') \right| \\ &\leq \int_H \left| \, h(w' + \frac{(x_1+s)}{u} y') - h(w' + \frac{(x_1)}{u} y') \right| \mathcal{N}(0, Q')dy' \; \leq c_3 [h]_\theta \, \frac{s^\theta}{u^\theta}. \end{aligned} \tag{5.3.27}$$

Now we can conclude, since

$$\Gamma_1 \; \leq \; c_3 [h]_\theta \int_0^\infty e^{-\frac{u^2}{2\lambda_1}} \, \frac{s^\theta}{u^\theta}\, du \; \leq c_4 \, [h]_\theta s^\theta, \;\; x_1, \; s \in \mathbb{R}_+, \; w' \in H'. \; ■ \tag{5.3.28}$$

Remark 5.3.14 Consider the problem:

$$\begin{cases} \lambda\phi(x) \; - \; \frac{1}{2}\phi''(x) \; = \; 1, \qquad x \in \mathbb{R}_+ \\ \phi(0) \; = \; 0 \end{cases} \tag{5.3.29}$$

From theorem 5.2.13, we know that $R(\lambda, \mathcal{T})1$ is the unique solution of the problem. Thus we have

$$R(\lambda, \mathcal{T})1 \; = \; \frac{1}{\lambda}(1 - e^{-\sqrt{\lambda}x}\,).$$

Moreover by formulas 5.2.29 and 5.2.30, we can get, for any $x \in \mathbb{R}_+$,

$$\begin{aligned} -e^{-\sqrt{\lambda}x} \; &= \phi''(x) = \; -\frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{x}{t\sqrt{t}} e^{\frac{-x^2}{2t}}\, e^{-\lambda t}\, dt \\ &= \; -\frac{4}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{u^2}{2}}\, e^{-\lambda\,\frac{x^2}{u^2}}\, du, \end{aligned} \tag{5.3.30}$$

that is useful, see (5.3.25).

Remark 5.3.15 If $H=\mathbb{R}^n$, $n\geq 1$ then Proposition 5.3.13 is a trivial consequence of Theorem 5.3.12; indeed if $\psi = R(\lambda,\mathcal{T})f$, $f\in \mathcal{C}_b^\theta(\mathbb{R}^n_+)$, $\lambda>0$, $\theta\in(0,1)$, then $\psi\in\mathcal{C}_b^{2+\theta}(\mathbb{R}^n_+)$ ([7]) so that ψ is a strict solution and $\mathcal{T}\psi=\frac{1}{2}\sum_{k=1}^n\lambda_k D_{kk}\psi$.

Now from the equality $D_{11}\psi = 2\mathcal{T}\psi - \sum_{k=2}^n\lambda_k D_{kk}\psi$, we can deduce Hölder regularity and Schauder estimate for $D_{11}\psi$.
Instead when dim $H=\infty$, for a datum $f\in\mathcal{C}_Q^\theta(H_+)$, it is not clear if

$$\mathcal{T}\psi(x)=\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k D_{kk}\psi(x),\quad x\in H_+.$$

Moreover we are not able to prove that $\sum_{k=2}^\infty\lambda_k D_{kk}\psi\in\mathcal{C}_Q^\theta(H_+)$. ∎

Using the spaces introduced at the beginning of this section, we present our final version of Schauder estimates, that improves Theorem 4.12 in Priola [66].

Theorem 5.3.16 *Consider* $\psi = R(\lambda,\mathcal{T})g$, $g\in\mathcal{N}_Q^\theta(H_+)$ $\lambda>0$, $\theta\in(0,1)$. *Then* $\psi\in\mathcal{C}_Q^2(H_+)$ *and* $D_Q^2\psi\in\mathcal{N}_Q^\theta(H_+,\mathcal{L}_2(H))$. *Moreover there exists a constant* $c=c(\theta,Q,\lambda)>0$ *such that:*

$$\|\psi\|_{2,Q}+\|D_Q^2\psi\|_{\theta,\mathcal{N},\mathcal{L}_2}+\lambda\|\psi\|_{\theta,\mathcal{N}}+\|\mathcal{T}\psi\|_{\theta,\mathcal{N}}\leq c\,\|g\|_{\theta,\mathcal{N}}.$$

Proof We collect all previous results on Schauder estimates.
We only verify that $D_Q^2\psi(0,\cdot)$ is θ-Hölder continuous. Clearly we have $D_Q^2\psi(0,x')=D_{11}\psi(0,x')$ for any $x'\in H'$.
Introducing $f\in\mathcal{C}_0(H_+)$ and $h\in\mathcal{C}_b(H')$, $f(x)=g(x)-g(0,x')$, $x\in H_+$, and $h(x')=g(0,x')$, $x'\in H'$, we have
$\psi=\psi_1+\psi_2$, where $\psi_1=R(\lambda,\mathcal{T})f$, $\psi_2=R(\lambda,\mathcal{T})h$.
Now by Theorem 5.3.5, we know that $D_{11}\psi_1(0,x')=0$, $x'\in H'$. Thus, applying Proposition 5.3.13, we find that $D_{11}\psi_2(0,\cdot)\in\mathcal{C}_b^\theta(H')$ and the assertion follows. ∎

[7] $\mathcal{C}_b^{2+\theta}(\mathbb{R}^n_+)\overset{\text{def}}{=}\{f\in\mathcal{C}_b^2(\mathbb{R}^n_+)$ such that $D^2f:\mathbb{R}^n_+\to\mathcal{L}(\mathbb{R}^n)$ is θ-Hölder continuous $\}$.

Part III

Parabolic equations in infinite dimensions

Chapter 6

On a class of Markov type semigroups in uniformly continuous and bounded functions spaces

Here we study a new class of Markov-type semigroups (not strongly continuous in general) in the space of all real, uniformly continuous and bounded functions on a separable metric space E. We call these semigroups, $\pi-$semigroups. Our results allow us to characterize the generators of Markov transition semigroups in infinite dimensions associated with (0.1.1) and (0.1.2), as for instance the Ornstein-Uhlenbeck semigroups.

6.1 Introduction

In this chapter we study a new class of semigroups of bounded linear operators on $\mathcal{C}_b(E)$, the Banach space of all real, uniformly continuous and bounded functions on a separable metric space E, endowed with the supremum norm $\|\cdot\|_0$. Following Priola [67], [68], [69], we call these semigroups, π-semigroups. A π-semigroup P_t is characterized by the following assumptions:

(i) for any $f \in \mathcal{C}_b(E)$, $x \in E$, the map $[0,\infty[\rightarrow \mathbb{R}$, $t \rightarrow P_t f(x)$ is continuous;

(ii) for any bounded sequence $(f_n) \subset \mathcal{C}_b(E)$ such that f_n converges pointwise to $f \in \mathcal{C}_b(E)$ (we briefly write that $f_n \xrightarrow{\pi} f$), we have $P_t f_n \xrightarrow{\pi} P_t f$, $t \geq 0$;

(iii) there exist $M \geq 1$ and $\omega \geq 0$ such that $\|P_t f\|_0 \leq M\, e^{\omega t}\, \|f\|_0$, $f \in \mathcal{C}_b(E)$, $t \geq 0$.

The main motivation is the study of semigroups of kernels in infinite dimensions. They arise as transition semigroups of Markov processes (see Definition 6.2.15) corresponding to the solutions of stochastic differential equations and representing the solutions of PDE's with infinitely many variables, see (0.1.1) and (0.1.2). These semigroups, when considered as a family of operators acting on $\mathcal{C}_b(\Omega)$, where Ω is an open set of a separable Hilbert space H, turn out to be π-semigroups, see Section 6.3 and also Zambotti [87]. On the other hand in several cases the strong continuity fails to hold in $\mathcal{C}_b(\Omega)$. This happens for instance for the Ornstein-Uhlenbeck semigroup, associated with equations (0.1.1) and (0.1.2) when $Q(x) = Q$, $x \in H$, or for the semigroup associated with a Dirichlet problem in a half space of H, see Chapter 5.

The content of the chapter is organized into four parts. In Section 6.2 we deal with general properties of $\pi-$semigroups. In Section 6.3 we discuss some "concrete" examples. Section 6.4 is devoted to characterizing the generator of a $\pi-$semigroup, by proving a Hille-Yosida type theorem. Finally in the last section we show that the theory of π-semigroups can be also developed on $\mathcal{BC}(E)$, the Banach space of all real continuous and bounded functions on E, endowed with the sup norm. Notice that many transition Markov semigroups, even the heat semigroup, are not strongly continuous on $\mathcal{BC}(H)$ (on this subject see Tessitore and Zabczyk [76]).

In §6.2 we construct a theory of π-semigroups parallel to that of $\mathcal{C}_0-$semigroups on $\mathcal{C}_b(E)$. We define a generator $\mathcal{A}$ for a π-semigroup P_t on $\mathcal{C}_b(E)$ as follows:

$$\begin{cases} D(\mathcal{A}) = \{f \in \mathcal{C}_b(E) :\ \exists g \in \mathcal{C}_b(E),\ \exists \delta > 0 \text{ such that } \sup\limits_{h \in]0,\delta]} \|\Delta_h f\|_0 < \infty \\ \text{and } \lim\limits_{h \to 0^+} \Delta_h f(x) = g(x),\ \ x \in E\}, \\ \mathcal{A}f(x) \ = \lim\limits_{h \to 0^+} \Delta_h f(x), \quad f \in D(\mathcal{A}),\ x \in E, \end{cases}$$

where $\Delta_h = h^{-1}(P_h - I)$. We show that $\mathcal{A}$ is a closed operator in $\mathcal{C}_b(E)$ but does not have dense domain in general, see Proposition 6.2.9. Moreover the resolvent operator of $\mathcal{A}$ can be obtained by a Laplace transform of P_t, which is pointwise defined in $\mathcal{C}_b(E)$, see Proposition 6.2.11.

In this section there are two main results, that we briefly present here. Let P_t be a π-semigroup and let $\mathcal{S}$ be any covering of E. We consider the following operator:

$D(\mathcal{A}_\mathcal{S}) = \{f \in D(\mathcal{A}) \text{ such that } \lim_{h\to 0^+} \sup_{x\in S} |\Delta_h f(x) - \mathcal{A}f(x)| = 0,\ S \in \mathcal{S}\}$,
$\mathcal{A}_\mathcal{S} f(x) = \mathcal{A}f(x),\ \ x \in E$.

Our first main theorem (see Theorem 6.2.13) is a kind of generalization of a well know result that states that for a $\mathcal{C}_0-$semigroup, the "weak" and the "strong" generators coincide (see for instance Theorem 1.3 in Pazy [61]). It asserts that if

$$\lim_{t\to 0^+} \sup_{x\in S} |P_t f(x) - f(x)| = 0, \quad f \in \mathcal{C}_b(E),\ \ S \in \mathcal{S}, \tag{6.1.1}$$

then we have $\mathcal{A}_\mathcal{S} = \mathcal{A}$. As an useful corollary, see Corollary 6.2.14, by taking $\mathcal{S} = \{H\}$, we obtain that if a π-semigroup P_t is also a $\mathcal{C}_0-$semigroup on $\mathcal{C}_b(E)$, then the generators of P_t as $\mathcal{C}_0-$semigroup and as π-semigroup coincide. The second result concerns the existence of a locally convex topology on $\mathcal{C}_b(E)$, which induces the π-convergence for sequences, see Theorem 6.2.3.

The theory of π-semigroups is a development of Cerrai's theory of weakly continuous semigroups, see Cerrai [14], Cerrai and Gozzi [15] and Remark 6.2.2. They were introduced to study the Ornstein-Uhlenbeck semigroup on $\mathcal{C}_b(H)$, whose generator was defined through the pointwise Laplace transform of the semigroup. The same approach has been used to define a generator for other semigroups such as the Mehler semigroups (see Fuhrman and Röckner [36]) and the semigroup arising from an infinite dimensional Dirichlet problem (see Chapter 5, Proposition 5.2.8). In Section

6.3 we show that all these semigroups are in fact π-semigroups and that their generators can be also defined through a pointwise limit of $\Delta_h f$ or equivalently through a uniform limit of $\Delta_h f$ on each compact set.

Since π-semigroups are not strongly continuous in general, a comparison with other types of semigroups seems to be in order. A way to treat the lack of strong continuity for a semigroup is to find a suitable linear locally convex topology weaker than the norm topology of the underlying Banach space but more appropriate for the semigroup. Let us remark that the classical Yosida approach (see §IX.3 of Yosida [88]) in the treatment of semigroups of linear operators on locally convex spaces does not work in our case. Indeed it requires that the locally convex topologies are sequentially complete (see claim 4 of Theorem 6.2.3).

Several papers about semigroups on general locally convex spaces are available in the literature (see Jefferies [44], [45] and the references therein). On this subject we can show that π-semigroups are weakly integrable semigroups in the Jefferies sense, see Remark 6.2.17. However our approach is different and simpler. In order to treat π-semigroups, we do not use weak Pettis-type integration and do not have to consider the properties of a particular locally convex topology on $\mathcal{C}_b(E)$, which is difficult to characterize (see Theorem 6.2.3). We will work in $\mathcal{C}_b(E)$ only with respect to the norm topology.

We consider the connections with the class of integrated semigroups, which has been intensively studied (see for instance Arendt [3], Hieber and Kellerman [43], Thieme [77]). We point out that any generator of a π-semigroup is the generator of an integrated semigroup on $\mathcal{C}_b(E)$ as well (see Proposition 6.2.12). However our results do not follow from the general theory of integrated semigroups.

Finally one can consider analytic semigroups T_t on a Banach space X (i.e. the map $t \mapsto T_t$ is analytic in $]0,\infty[$ with values in $\mathcal{L}(X)$) without requiring the strong continuity at $t = 0$. This theory is developed in the book [55] by Lunardi to treat sistematically parabolic PDE's in finite dimensions. Unfortunately the Ornstein-Uhlenbeck semigroup is not analytic even in $\mathcal{C}_b(\mathbb{R}^n)$ (see Da Prato and Lunardi [21]). In infinite dimensions the situation is worse: even the heat semigroup is not analytic in $\mathcal{C}_b(H)$ (see Guiotto [42] and also Chapter 3).

6.2 Basic properties of π-semigroups

6.2.1 Preliminaries

Let (E, d) be a separable metric space, with metric d, we denote by $\mathcal{C}_b(E)$ the set of all real, uniformly continuous and bounded functions on E. We consider $\mathcal{C}_b(E)$ as a Banach space endowed with the sup norm:

$$\|f\|_0 = \sup_{x\in E} |f(x)|, \quad f \in \mathcal{C}_b(E).$$

A sequence $(f_n) \subset \mathcal{C}_b(E)$ is said to be π- *convergent* to a map f and we shall write $f_n \xrightarrow{\pi} f$ if the following conditions hold:

$$
\begin{aligned}
&(a) \quad f \in \mathcal{C}_b(E), \qquad \sup_{n\geq 1} \|f_n\|_0 \;<\; \infty; \\
&(b) \quad \lim_{n\to\infty} f_n(x) \;=\; f(x), \quad x \in E.
\end{aligned}
\tag{6.2.1}
$$

Similarly, let J be a real interval and $\hat{t} \in J$. Let $F : J\backslash\{\hat{t}\} \to \mathcal{C}_b(E)$, we say that

$$
F(t) \xrightarrow{\pi} f \;\text{ as }\; t \to \hat{t}, \tag{6.2.2}
$$

if for any sequence $(t_n) \subset J\backslash\{\hat{t}\}$ that converges to $\hat{t}$, we have that $F(t_n) \xrightarrow{\pi} f$.

Notice that the previous condition implies that there exists a neighborhood U of $\hat{t}$ such that $\sup_{t\in U\backslash\{\hat{t}\}} \|F(t)\|_0 < \infty$.

We emphasize that the notion of pointwise convergence for uniformly bounded sequences of functions is usual in the Theory of Markov Processes (we mention Ethier and Kurtz [33] (page 111), Dynkin [28], see also Definition 6.2.15).

Now we are ready to introduce π-semigroups on $\mathcal{C}_b(E)$.

Definition 6.2.1 Let $P_t,\, t \geq 0$ be a semigroup of bounded linear operators on $\mathcal{C}_b(E)$, namely $P_{t+s} = P_t\, P_s$, $P_0 = I$ for $t,\, s \geq 0$. We say that P_t is a *π-semigroup* on $\mathcal{C}_b(E)$ if the following conditions hold ([1]):

(i) there exist $M \geq 1$ and $\omega \geq 0$, such that $\|P_t\|_{\mathcal{L}(\mathcal{C}_b(E))} \leq M\, e^{\omega t}$, $t \geq 0$;

(ii) for any $x \in E$, $f \in \mathcal{C}_b(E)$, the map $[0,\infty[\to \mathbb{R}$, $t \mapsto P_t f(x)$ is continuous;

(iii) for any $(f_n) \subset \mathcal{C}_b(E)$, $f_n \xrightarrow{\pi} f$ implies that $P_t f_n \xrightarrow{\pi} P_t f$ as $n \to \infty$, $t \geq 0$. (6.2.3)

Let us remark that condition (i) is equivalent to require that the semigroup P_t is locally bounded (i.e. for any $T > 0$, there exists a constant C_T such that $\|P_t\|_{\mathcal{L}(\mathcal{C}_b(E))} \leq C_T$, $t \in [0,T]$). The proof is standard.

Let P_t be a π-semigroup, we define the *type* of P_t as the real number
$\omega = \inf\{\alpha \geq 0 \text{ such that there exists } M_\alpha \geq 1,\; \|P_t\|_{\mathcal{L}} \leq M_\alpha\, e^{\alpha t},\; t \geq 0\}$.

Let now $\mathcal{S} = \{S_i\}_{i\in I}$ be a non trivial covering of E, i.e. $S_i \subset E$, $i \in I$, $E = \cup_{i\in I} S_i$ and assume that there exists $S_{\hat{i}} \in \mathcal{S}$ that is infinite. In the sequel we also consider π-semigroups P_t that satisfy the following additional condition:

$$
\lim_{t\to 0^+} \sup_{x\in S} |P_t f(x) - f(x)| \;=\; 0, \quad f \in \mathcal{C}_b(E),\;\; S \in \mathcal{S} \tag{6.2.4}
$$

Let us notice that in case P_t satisfies (6.2.4) with $\mathcal{S} = \{E\}$ then it is also a strongly continuous semigroup on $\mathcal{C}_b(E)$. ∎

Remark 6.2.2 π-Semigroups are a development of Cerrai's weakly continuous semigroups, see Cerrai [14], Cerrai and Gozzi [15].

[1]Let $(X, \|\cdot\|_X)$ be a real Banach space, we denote by $(\mathcal{L}(X), \|\cdot\|_{\mathcal{L}})$ the Banach space of all linear and continuous operators on X, endowed with the usual norm $\|T\|_{\mathcal{L}} = \sup_{\|x\|_X\leq 1} \|Tx\|_X$, $T \in \mathcal{L}(X)$.

Let H be a real separable Hilbert space, a sequence $(f_n) \subset \mathcal{C}_b(H)$ is said to be $\mathcal{K}$-convergent to $f \in \mathcal{C}_b(H)$, using the Cerrai notation, if for any compact subset $K \subset H$, we have that

$$\sup_{n\geq 1} \|f_n\|_0 < \infty, \quad \text{and} \quad \lim_{n\to\infty} \sup_{x\in K} |f_n(x) - f(x)| \,=\, 0.$$

Whereas weakly continuous semigroups preserve $\mathcal{K}$-convergence of sequences of functions, π-semigroups preserve π-convergence. Moreover any transition Markov semigroup (see Definition 6.2.15 for a precise definition) which is weakly continuous in Cerrai's sense is clearly a π-semigroup.

One may wonder why we do not require that a π-semigroup P_t also satisfies the following assumptions (compare with Definition 2.1, hypothesis 1 and 3, in Cerrai [14]):

$$\begin{aligned}
&(iii') \text{ for any } \ (f_n) \subset \mathcal{C}_b(E), \ \ f_n \xrightarrow{\pi} f \ \text{ implies that for any } x \in E \\
&\lim_{n\to\infty} P_t f_n(x) = P_t f(x) \ \text{ uniformly in } t \text{ on each bounded set of } [0,\infty[; \\
&(iv) \text{ for any } f \in \mathcal{C}_b(E), \text{ the family of maps } \ \{P_t f\}_{t\geq 0} \ \text{ is equi-uniformly continuous.}
\end{aligned} \tag{6.2.5}$$

To this purpose we provide two simple examples of π- semigroups for which (iii') or (iv) do not hold.

1. Consider the following semigroup for $t \geq 0$:

$$T_t : \mathcal{C}_b(\mathbb{R}) \,\to\, \mathcal{C}_b(\mathbb{R}), \quad T_t f(x) = \, f(x+t), \ \ f \in \mathcal{C}_b(\mathbb{R}), \ x \in \mathbb{R}. \tag{6.2.6}$$

T_t is clearly a π-semigroup on $\mathcal{C}_b(\mathbb{R})$ (it is also a strongly continuous semigroup) but it does not satisfy condition (iii').

Indeed take a sequence $(f_n) \subset \mathcal{C}_b(\mathbb{R})$ defined as follows

$$f_n(x) \,=\, \frac{nx}{1+n^2x^2}, \qquad x \in \mathbb{R}, \ \ n \geq 1,$$

we have that $\lim_{n\to\infty} f_n(x) = 0$, $x \in \mathbb{R}$ and $\sup_{n\geq 1} \|f_n\|_0 = \frac{1}{2}$.
However fix any closed interval $[a,b] \subset [0,\infty[$, and take $\hat{x} = -a$. We have that the sequence of maps: $t \to T_t f_n(-a) = f_n(-a+t)$, $n \geq 1$, does not converge to 0 uniformly in $t \in [a,b]$.

2. Consider the following semigroup for $t \geq 0$:

$$S_t : \mathcal{C}_b(\mathbb{R}) \,\to\, \mathcal{C}_b(\mathbb{R}), \quad S_t f(x) = \, e^{-t/2}\, f(e^t x), \ \ f \in \mathcal{C}_b(\mathbb{R}), \ x \in \mathbb{R}. \tag{6.2.7}$$

S_t is clearly a π-semigroup, but it does not satisfy condition (iv).

To see this fact, let us take $\hat{f}(x) = \min(1, |x|)$, $x \in \mathbb{R}$ and denoting by $\omega_g(s)$, $s \geq 0$ the modulus of continuity of a map $g \in \mathcal{C}_b(\mathbb{R})$, we have:

$$\omega_{S_t\hat{f}}(s) \,=\, \sup_{|x-y|\leq s} \, e^{-t/2}|\hat{f}(e^t x) - \hat{f}(e^t y)| \,=\, s\, e^{t/2}, \ \ t \geq 0. \ \blacksquare$$

Clearly $\pi-$convergence of sequences of functions does not define a unique topology on $\mathcal{C}_b(E)$. However the next result introduces a "natural" topology on $\mathcal{C}_b(E)$ associated with π-semigroups.

We fix some notations. Let μ be a Borel signed and finite measure on (E, d), by the Hahn-Jordan Decomposition Theorem, we can set $\mu = \mu_+ - \mu_-$, where μ_+ and μ_- are positive Borel measures, respectively the positive and negative part of μ. Moreover the variation of μ is the positive Borel measure $|\mu| = \mu_+ + \mu_-$ (we refer to Chapter 6 in Aliprantis and Burkinshaw [2] for details). Finally for any $B \subset E$, we denote by I_B the indicator of B (i.e. $I_B(x) = 0$ if $x \notin B$, $I_B(x) = 1$ if $x \in B$.)

Theorem 6.2.3 *There exists a Hausdorff locally convex topology τ_0 on $\mathcal{C}_b(E)$ such that it holds:*
(*) for any $(f_n) \subset \mathcal{C}_b(E)$, f_n converges to $f \in \mathcal{C}_b(E)$ with respect to τ_0 $\Leftrightarrow f_n \xrightarrow{\pi} f$.
τ_0 is not metrizable and not sequentially complete.

Proof We consider $\mathcal{M}(E)$, the linear space of all Borel signed and finite measures on E. It is straightforward to verify that $\mathcal{M}(E)$ is a Banach space endowed with the norm $\|\mu\|_{\mathcal{M}} = |\mu|(E)$, $\mu \in \mathcal{M}(E)$, see Aliprantis and Burkinshaw [2].

We shall show first that $(\mathcal{M}(E), \|\cdot\|_{\mathcal{M}})$ is isometrically embedded in $\mathcal{C}_b(E)'$, the topological dual of $\mathcal{C}_b(E)$, endowed with the dual norm $\|\cdot\|'$ and then that $\tau_0 = \sigma(\mathcal{C}_b(E), \mathcal{M}(E))$ ([2]) is the topology looked for. The proof is split up into several parts.

Claim 1. $(\mathcal{M}(E), \|\cdot\|_{\mathcal{M}})$ *is isometrically embedded in* $(\mathcal{C}_b(E)', \|\cdot\|')$.

Consider the map $F : \mathcal{M}(E) \to \mathcal{C}_b(E)'$, such that for any $\mu \in \mathcal{M}(E)$, F_μ is defined by the formula:

$$< F_\mu, f >= \int_E f(y)\, \mu(dy), \quad \mu \in \mathcal{M}(E), \quad f \in \mathcal{C}_b(E).$$

We assert that F is an isometry. It is evident that $\|F_\mu\|' \le \|\mu\|_{\mathcal{M}}$ for any $\mu \in \mathcal{M}(E)$, let us prove the converse inequality.

To this purpose fix a $\mu \in \mathcal{M}(E)$, $\mu = \mu_+ - \mu_-$. There exist two Borel sets A_+ and A_- such that $A_+ \cap A_- = \emptyset$, $A_+ \cup A_- = E$ and further $\mu_+(A_+) = \mu_+(E)$, $\mu_-(A_-) = \mu_-(E)$.

Fix $\epsilon > 0$, by a property of Borel finite measures (see for instance Theorem 4.3.7 of Ash [4]), we can choose a closed set of E: $C_- \subset A_-$ such that $\mu_-(A_- \backslash C_-) < \epsilon$. Now the crucial point of the proof consists in showing that there exists a closed set of E: $C_+ \subset A_+$ such that

$$\mu_+(A_+ \backslash C_+) < \epsilon \text{ and further } C_+ \text{ and } C_- \text{ are separated} \tag{6.2.8}$$

(i.e. $d(C_+, C_-) = \inf_{x \in C_+, y \in C_-} d(x, y) > 0$). We start to take a closed set $C \subset A_+$ such that $\mu_+(A_+ \backslash C) < \epsilon/2$. Then we consider a sequence of closed sets defined as follows, $C_n = \{x \in C \text{ such that } d(x, C_-) \ge n^{-1}\}$, $n \ge 1$.

[2]Let E be a Banach space and F a subspace of E'. The $\sigma(E, F)$ topology is the weakest topology on E making each $\eta \in F$ continuous. It is a locally convex topology (see for instance §V.3.2 in Dunford and Schwarz [29])

Now we prove that $\mu_+(B_n) \to 0$ as $n \to \infty$, where $B_n = C\backslash C_n$. We have that $B_n \downarrow B_0$ (i.e. $B_{n+1} \subset B_n$ and $\cap_{n\geq 1} B_n = B_0$). Let us remark that since $C \cap C_-$ $= \emptyset$ we get that $B_0 = \emptyset$ and so $\mu_+(C\backslash C_n) \to 0$ as $n \to \infty$. Hence there exists n_0 such that $\mu_+(C\backslash C_{n_0}) < \epsilon/2$. Thus (6.2.8) is proved by setting $C_+ = C_{n_0}$. Indeed $d(C_{n_o}, C_-) \geq {n_0}^{-1}$ and

$$\mu_+(A_+\backslash C_{n_0}) \leq \mu_+(A_+\backslash C) + \mu_+(C\backslash C_{n_0}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Now consider the Borel map $g = I_{A_+} - I_{A_-}$. It is clear that

$$\int_E g(y)\mu(dy) = \int_E g(y)\mu_+(dy) - \int_E g(y)\mu_-(dy) = |\mu|(E).$$

Since C_+ and C_- are separated closed sets we can take a map $f \in \mathcal{C}_b(E)$ such that $\|f\|_0 = 1$, $f(x) = 1$ if $x \in C_+$ and $f(x) = -1$ if $x \in C_-$ (for instance set $f(x)$ $= [d(x, C_-) + d(x, C_+)]^{-1}\, [d(x, C_-) - d(x, C_+)]$, $x \in E$). We can verify that

$$|\int_E f(y)\,\mu(dy) - \int_E g(y)\,\mu(dy)| \leq \int_E |f(y) - g(y)|\,|\mu|(dy)$$

$$= \int_{A_+\backslash C_+} |f(y) - g(y)|\,\mu_+(dy) + \int_{A_-\backslash C_-} |f(y) - g(y)|\,\mu_-(dy) \leq 4\epsilon.$$

Therefore $\|F_\mu\|' \geq \, < F_\mu, f > \, \geq \int_E g(y)\mu(dy) - 4\epsilon = |\mu|(E) - 4\epsilon$. For the arbitrariness of ϵ we conclude that $\|F_\mu\|' \geq \|\mu\|_{\mathcal{M}}$. Thus F is an isometry.

Claim 2. $\tau_0 = \sigma(\mathcal{C}_b(E), \mathcal{M}(E))$ *satisfies condition (*).*

For any $x \in E$ we denote by δ_x the Dirac measure with support $\{x\}$. Let us notice that τ_0 is a Hausdorff topology, since Dirac measures separate the elements of $\mathcal{C}_b(E)$.

We prove property (*).

$\Leftarrow$ It is clear, using the Dominated Convergence Theorem.

$\Rightarrow$ If $f_n \to f$ with respect to τ_0, then using the Dirac measures we immediately conclude that $\lim_{n\to\infty} f_n(x) = f(x)$, $x \in E$.

Assume by contradiction that $\sup_{n\geq 1} \|f_n\|_0 = \infty$. We can suppose that $\lim_{n\to\infty} f_n(x)$ $= 0$, $x \in E$ and that $f_n \geq 0$ (using that $f_n(x) = \max(f_n(x), 0) - \max(-f_n(x), 0)$, $x \in E$).

Since (f_n) is not equibounded, there exists a subsequence, denoted by (f_k), and a sequence of points $(x_k)_{k\geq 1} \subset E$, such that $f_k(x_k) > 2^k$, $k \geq 1$. Now consider the measure μ_0, $\mu_0(B) = \sum_{k=1}^{\infty} 2^{-k}\delta_{x_k}(B)$, for any Borel set B in E. It is simple to verify that $\mu_0 \in \mathcal{M}(E)$. Moreover we have for any $k \geq 1$:

$$\int_E f_k(y)\mu_0(dy) \geq 2^{-k} \int_E f_k(y)\delta_{x_k}(dy) = 2^{-k} f_k(x_k) \geq 1.$$

Thus f_k can not converge to 0 with respect to τ_0 and we have obtained a contradiction. The claim is proved.

Claim 3. τ_0 *is not metrizable.*

Actually it is possible to prove that τ_0 does not satisfy the first countable axiom, even if $E = \mathbb{R}$. We use the following theorem: if X is a Banach space and the topology $\sigma(X, X')$ satisfies the first countable axiom then X has finite dimension. For the proof we refer the reader to § II, pag. 10 of Diestel [28].

Let us remark that the previous result also holds, with the same proof, if the topology $\sigma(X, X')$ is replaced by the one $\sigma(X, Y)$, where Y is a closed subspace of X'. Now to conclude we observe that, by claim 1, using the isometry F, $\mathcal{M}(E)$ can be considered as a closed subset of $\mathcal{C}_b(E)'$.

Claim 4. τ_0 *is not sequentially complete.*

Actually we are able to prove a stronger statement: *any locally convex topology τ on $\mathcal{C}_b(E)$, satisfying condition (*) (with τ_0 replaced by τ) is not sequentially complete.*

Denote by Γ the family of all seminorms on $\mathcal{C}_b(E)$ which are continuous with respect to τ. We recall that a sequence $(f_n) \subset \mathcal{C}_b(E)$ is a $\tau-$Cauchy sequence if for any $\epsilon > 0$, $q \in \Gamma$, there exists N such that for any $n \geq N$ and $m \geq N$ it holds: $q(f_n - f_m) < \epsilon$. This is equivalent to ask that

$$\lim_{n\to\infty} \sup_{k\geq 1} q(f_{n+k} - f_n) = 0, \quad \text{for any} \quad q \in \Gamma.$$

We fix $a \in E$ and introduce the following functions: $\hat{f}_n(x) = \exp[-nd(x,a)]$, $x \in E$, $n \geq 1$. It is clear that $(\hat{f}_n) \subset \mathcal{C}_b(E)$, we prove that it is a $\tau-$Cauchy sequence.

We argue by contradiction. Assume that $(\hat{f}_n)$ is not a $\tau-$Cauchy sequence. Then there exists $\epsilon > 0$, $\hat{q} \in \Gamma$, and a sequence of integers (k_n) such that $\hat{q}(\hat{f}_{n+k_n} - \hat{f}_n) \geq \epsilon$ for any $n \geq 1$. From this inequality we deduce that the sequence $\hat{g}_n = \hat{f}_{n+k_n} - \hat{f}_n$ does not converge to $0 \in \mathcal{C}_b(E)$ with respect to τ.

We obtain a contradiction by showing that $\hat{g}_n$ τ-converge to 0 as $n \to \infty$. Notice that in virtue of condition (*), $\hat{g}_n$ τ-converge to 0 if and only if $\hat{g}_n \xrightarrow{\pi} 0$ as $n \to \infty$. Now since

$$\hat{g}_n(x) = e^{-n\,d(x,a)}\,(e^{-k_n\,d(x,a)} - 1), \;\; x \in E, \; n \geq 1,$$

we have that $\|\hat{g}_n\|_0 \leq 1$, $n \geq 1$. Let us consider the pointwise convergence of $\hat{g}_n$. If $x = a$, then $\hat{g}(a) = 0$ for any $n \geq 1$. If $x \neq a$, since $|\hat{g}_n(x)| \leq \exp[-n\,d(x,a)]$, $n \geq 1$, we find that $\lim_{n\to\infty} \hat{g}_n(x) = 0$. Therefore $\hat{g}_n \xrightarrow{\pi} 0$ as $n \to \infty$. It follows that $(\hat{f}_n)$ is a $\tau-$Cauchy sequence.

Now let us notice that $\lim_{n\to\infty} \hat{f}_n(x) = I_{\{a\}}(x)$, $x \in E$. Thus $(\hat{f}_n)$ can not π-converge to a map in $\mathcal{C}_b(E)$. Consequently τ is not sequentially complete.
The proof is complete. ∎

Given a π-semigroup P_t on $\mathcal{C}_b(E)$, by the previous proposition we deduce that P_t is semigroup of linear operators, which are sequentially continuous on $\mathcal{C}_b(E)$ with respect to τ_0.

In this paper we will not investigate if the operators P_t, $t \geq 0$ are actually τ_0-continuous on $\mathcal{C}_b(E)$ or not. Thus we only consider on $\mathcal{C}_b(E)$ the sup norm topology.

Let us consider the connections between π-semigroups and strongly continuous semigroups. It is clear, by the Dominated Convergence Theorem, that any strongly continuous and transition Markov semigroup (see Definition 6.2.15) is in particular a π-semigroup. However in general the class of strongly continuous semigroups and that of π-semigroups are different as the next result shows.

Proposition 6.2.4 *There exists a uniformly continuous semigroup S_t on $\mathcal{C}_b(\mathbb{R})$ that is not a π-semigroup.*

Proof Denote by $\mathcal{C}_b(\mathbb{R})^{'}$ the topological dual of $\mathcal{C}_b(\mathbb{R})$, and by $< \cdot, \cdot >$ the duality between the previous spaces. The proof is split up into two parts.
Claim 1. There exists $\hat{G} \in \mathcal{C}_b(\mathbb{R})^{'}$ and a sequence $(\hat{u}_n) \subset \mathcal{C}_b(\mathbb{R})$ such that $\hat{u}_n \xrightarrow{\pi} \hat{u}$ as $n \to \infty$ but $< \hat{G}, \hat{u}_n >$ does not converge to $< \hat{G}, \hat{u} >$.

We argue by contradiction. Suppose that the claim 1 fails to hold, then any $F \in \mathcal{C}_b(\mathbb{R})^{'}$ preserves the $\pi-$ convergence for sequences of $\mathcal{C}_b(\mathbb{R})$. Fix a π-semigroup P_t on $\mathcal{C}_b(\mathbb{R})$ and a map $f \in \mathcal{C}_b(\mathbb{R})$. By the properties of P_t, we have $P_t f \xrightarrow{\pi} f$ as $t \to 0^+$. Hence for any sequence $(t_n) \subset \mathbb{R}_+$ such that $t_n \to 0^+$, we get

$$\lim_{n\to\infty} < P_{t_n} f, F > = < f, F >, \quad F \in \mathcal{C}_b(\mathbb{R})^{'}.$$

Thus we get $\lim_{t\to 0^+} P_t f = f$ with respect to $\sigma(\mathcal{C}_b(\mathbb{R}), \mathcal{C}_b(\mathbb{R})^{'})$.

Invoking a well known result (see Section IX.1 in Yosida [88], " weak equal strong") we will deduce that P_t is a strongly continuous semigroup on $\mathcal{C}_b(\mathbb{R})$. But this is not true, since P_t is an arbitrary π-semigroup and many π-semigroups on $\mathcal{C}_b(\mathbb{R})$, for instance the Ornstein-Uhlenbeck semigroup (see Section 6.3), are not strongly continuous. Thus claim 1 is proved.
Claim 2. Construction of S_t.

We will use $\hat{G} \in \mathcal{C}_b(\mathbb{R})'$ and $(\hat{u}_n) \subset \mathcal{C}_b(\mathbb{R})$ as in claim 1. Fix $\hat{g} \in \mathrm{Ker}\hat{G}$, $\hat{g} \neq 0$ and define the operator

$$T : \mathcal{C}_b(\mathbb{R}) \to \mathcal{C}_b(\mathbb{R}), \quad Tf(x) = < \hat{G}, f > \hat{g}(x), \quad f \in \mathcal{C}_b(\mathbb{R}),\ x \in \mathbb{R}.$$

It is evident that $T \in \mathcal{L}(\mathcal{C}_b(\mathbb{R}))$. Further we have $T^k f = < \hat{G}, f >< \hat{G}^{k-1}, \hat{g} > \hat{g}$, $k \geq 1$. Hence $T^k f = 0$ for any $f \in \mathcal{C}_b(\mathbb{R})$, $k \geq 2$.
Define S_t, $t \geq 0$, as

$$S_t f = \sum_{k=0}^{\infty} \frac{t^k T^k f}{k!} = f + tTf = f + < \hat{G}, f > \hat{g}, \quad f \in \mathcal{C}_b(\mathbb{R}).$$

We state that, for any $t \geq 0$, it does not hold: $S_t \hat{u}_n \xrightarrow{\pi} S_t \hat{u}$ as $n \to \infty$.
To this purpose, fix $t > 0$ and choose $x_0 \in \mathbb{R}$ such that $\hat{g}(x_0) \neq 0$. We obtain:

$$S_t \hat{u}_n(x_0) = \hat{u}_n(x_0) + t < \hat{G}, \hat{u}_n > \hat{g}(x_0) \text{ and}$$
$$S_t \hat{u}(x_0) = \hat{u}(x_0) + t < \hat{G}, \hat{u} > \hat{g}(x_0).$$

Now it is clear that $\lim_{n\to\infty} \hat{G}(\hat{u}_n) = \hat{G}(\hat{u})$ if and only if $\lim_{n\to\infty} S_t \hat{u}_n(x_0) = S_t \hat{u}(x_0)$. Hence S_t is not a π-semigroup. The assertion is proved. ∎

We can modify the previous proof in order to obtain the existence of a uniformly continuous semigroup on $\mathcal{C}_b(\mathbb{R})$ that is not a weakly continuous semigroup in Cerrai's sense.

Indeed as in claim 1, we can prove that there exist $\hat{F} \in \mathcal{C}_b(\mathbb{R})^{'}$ and a sequence $(\hat{u}_n) \subset \mathcal{C}_b(\mathbb{R})$ such that u_n $\mathcal{K}-$converges to $u \in \mathcal{C}_b(\mathbb{R})$, as $n \to \infty$, but $< \hat{F}, \hat{u}_n >$ does not converge to $< \hat{F}, \hat{u} >$.

Remark 6.2.5
(a) Consider a semigroup T_t of bounded linear operators on $\mathcal{C}_b(E)$ having properties (i) and (iii) of Definition 6.2.1 and moreover satisfying

$$(ii') \text{ for any } x \in E,\ f \in \mathcal{C}_b(E), \quad \lim_{t\to 0^+} T_t f(x) \,=\, f(x).$$

One can easily obtain that for any $f \in \mathcal{C}_b(E)$ and $x \in E$, the map $\eta_{f,x} : [0, \infty[\,\to \mathbb{R}$, $t \mapsto P_t f(x)$ is right-continuous. It is not clear if (ii') implies assumption (ii) (i.e. if the map $\eta_{f,x}$ is continuous). We refer to Chapter XII of Dellacherie and Meyer [26], that is related to this question.

(b) In case E is also a *compact set*, in order that a semigroup P_t of bounded linear operators on $\mathcal{C}_b(E)$ is a π-semigroup and also a strongly continuous semigroup, it is enough that P_t satisfies the following two conditions:

$$\begin{aligned} &(i) \text{ there exist } M \geq 1,\ \omega \geq 0, \text{ such that } \|P_t\|_{\mathcal{L}(\mathcal{C}_b(E))} \leq M\, e^{\omega t}, \ \ t \geq 0; \\ &(ii') \text{ for any } x \in E,\ f \in \mathcal{C}_b(E), \quad \lim_{t\to 0^+} P_t f(x) \,=\, f(x). \end{aligned} \tag{6.2.9}$$

Suppose that conditions (i) and (ii') are satisfied. First we prove that P_t verifies condition (iii) of Definition 6.2.1. Fix any $t > 0$, then by a general result on weak topologies, the linear operator P_t turns out to be also continuous on $\mathcal{C}_b(E)$ endowed with the weak topology $\sigma(\mathcal{C}_b(E), \mathcal{C}_b(E)')$. By a Riesz theorem, we know that $\mathcal{C}_b(E)'$ can be isometrically identified with $\mathcal{M}(E)$ and so $\sigma(\mathcal{C}_b(E), \mathcal{C}_b(E)') = \sigma(\mathcal{C}_b(E), \mathcal{M}(E))$. Combining these results, with Theorem 6.2.3 we can deduce that P_t preserves the π-convergence of sequences in $\mathcal{C}_b(E)$ and so condition (iii) holds.

Now we prove that P_t satisfies condition (ii) of Definition 6.2.1. Notice that conditions (i) and (ii') of (6.2.9) and Theorem 6.2.3 imply that $\lim_{t\to 0^+} P_t f = f$ with respect to $\sigma(\mathcal{C}_b(E), \mathcal{M}(E))$. But $\sigma(\mathcal{C}_b(E), \mathcal{M}(E))$ coincides with $\sigma(\mathcal{C}_b(E), \mathcal{C}_b(E)')$ and so applying well known result (see Section IX.1 in Yosida [88], " weak equal strong") we will deduce that P_t is a strongly continuous semigroup on $\mathcal{C}_b(E)$. Thus in particular condition (ii) is satisfied.

Hence if E is a compact set, on $\mathcal{C}_b(E)$ the class of π-semigroups and that of strongly continuous semigroups are the same.

(c) Suppose that $E \subset X$, where X is another separable metric space. We emphasize that a semigroup P_t of bounded linear operators on $\mathcal{C}_b(E)$ can be a π-semigroup on $\mathcal{C}_b(E)$ but not on $\mathcal{C}_b(\overline{E})$ (for instance see the semigroup in §6.3.2). Notice that this is impossible for strongly continuous semigroups. ∎

6.2.2 The generator of a π-semigroup

Definition 6.2.6 Let P_t be a π-semigroup on $\mathcal{C}_b(E)$ we set

$$\Delta_h \,=\, \frac{P_h - I}{h}, \quad h > 0$$

and we define its *infinitesimal generator* $\mathcal{A}$ as follows

$$\begin{cases} D(\mathcal{A}) \,=\, \{f \in \mathcal{C}_b(E) \text{ such that } \exists g \in \mathcal{C}_b(E)\ ,\ \ \Delta_h f \stackrel{\pi}{\longrightarrow} g \text{ as } h \to 0^+\} \\ \\ \mathcal{A}f(x) \stackrel{\text{def}}{=} \lim\limits_{h\to 0^+} \Delta_h f(x), \quad f \in D(\mathcal{A}),\ x \in E. \end{cases} \tag{6.2.10}$$

Let now $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{C}_b(E) \ \to\ \mathcal{C}_b(E)$ be a linear operator, we say that $\mathcal{L}$ is a π-*closed operator* if for any $(f_n) \subset D(\mathcal{L})$, the following condition holds:

$$f_n \stackrel{\pi}{\longrightarrow} f \ \text{ and } \mathcal{L}f_n \stackrel{\pi}{\longrightarrow} g \quad \Rightarrow \quad f \in D(\mathcal{L}) \text{ and } \ \mathcal{L}f = g \tag{6.2.11}$$

A subset $C \subset \mathcal{C}_b(E)$ is said to be π-*dense* in $\mathcal{C}_b(E)$ if for any $f \in \mathcal{C}_b(E)$, there exists $(f_n) \subset C$, such that $f_n \stackrel{\pi}{\longrightarrow} f$. ∎

Let us consider some properties of generators of π-semigroups.

Proposition 6.2.7 *Let $\mathcal{A}$ be the generator of a π-semigroup P_t of type ω on $\mathcal{C}_b(E)$, then for any $f \in D(\mathcal{A})$, it holds:*
(i) $P_t f \in D(\mathcal{A})$ and $\mathcal{A}P_t f = P_t\mathcal{A}f$, $t \geq 0$;
(ii) for any $x \in E$, the map: $[0,\infty[\to \mathbb{R}$, $t \mapsto P_t f(x)$ is continuously differentiable and one has $\frac{d}{dt}P_t f(x) = P_t\mathcal{A}f(x)$, $t \geq 0$.

Proof (i) Fix $f \in D(\mathcal{A})$ and $t > 0$, there exists $K \geq 0$ and $\delta > 0$, such that $\|\Delta_h f\|_0 \leq K$ for any $h \in]0,\delta]$.

Then $\|P_t \Delta_h f\|_0 \leq MK\, e^{\omega t}$, $h \in]0,\delta]$ and for any $x \in E$,

$$\lim_{h\to 0^+} \Delta_h P_t f(x) \,=\, \lim_{h\to 0^+} P_t \Delta_h f(x) = \, P_t\mathcal{A}f(x).$$

Thus $P_t f \in D(\mathcal{A})$ and $\mathcal{A}P_t f \,=\, P_t\mathcal{A}f$.

(ii) Fix $f \in D(\mathcal{A})$, $x \in E$ and consider the map $t \mapsto P_t f(x)$.By the assumption on f, there exists the right derivative $\frac{d^+}{dt}P_t f(x) = P_t\mathcal{A}f(x)$ at any $t \geq 0$.

Let us notice that the function $t \mapsto P_t\mathcal{A}f(x)$ is continuous and so applying a well known lemma of Real Analysis (see for instance §2.1.2 in Pazy [61]), we get that $P_{(\cdot)}f(x)$ is differentiable and moreover

$$\frac{d}{dt}P_t f(x) = P_t\mathcal{A}f(x) = \, \mathcal{A}P_t f(x), \quad t \geq 0. \ \blacksquare$$

To proceed with the study of the generator of a π-semigroup, we need a preliminary lemma. It is basic for the treatment of π-semigroups in $\mathcal{C}_b(E)$.

Lemma 6.2.8 *Let (Y,μ) be a measurable space (μ is a finite, positive and complete measure). Let (X,d) be a separable metric space. Consider a function $F : Y \times X \to \mathbb{R}$ that satisfies the following conditions:*

(i) $F(\cdot,x)$ is a measurable mapping for any $x \in X$;

(ii) $F(y,\cdot)$ *is a uniformly continuous mapping, for* $y \in Y$ μ *- a.e.;*
(iii) *there exists* $g : Y \to \mathbb{R}$, μ*- integrable such that* $|F(y,x)| \le g(y)$, $x \in X$, $y \in Y$ μ *- a.e..*
Then the map $h : X \to \mathbb{R}$,

$$h(x) = \int_Y F(y,x)\mu(dy),\ x \in X \ \textit{ is uniformly continuous and bounded.}$$

Proof The boundedness of h is clear, since

$$|h(x)| \le \int_Y |F(y,x)|\mu(dy) \le \int_Y g(y)\mu(dy), \quad x \in E,$$

and also its continuity by the Dominated Convergence Theorem. Let us prove the uniform continuity of h. For any $n \ge 1$, we consider the set

$$A_n = \{(x,x') \in X \times X \ \text{ such that } \ d(x,x') \le \frac{1}{n}\}$$

To verify the assertion, we prove that

$$\lim_{n\to\infty} \sup_{(x,x')\in A_n} |h(x) - h(x')| \ = \ 0. \tag{6.2.12}$$

Let us choose for any $n \ge 1$, a countable dense set D_n in A_n (since $X \times X$ is separable) and for $y \in Y$ μ-a.e. we have:

$$\sup_{(x,x')\in A_n} |F(y,x) - F(y,x')| = \sup_{(x,x')\in D_n} |F(y,x) - F(y,x')|, \quad n \ge 1,$$

since for $y \in Y$ μ - a.e., $|F(y,\cdot) - F(y,\cdot)|$ is uniformly continuous on $X \times X$.

Now remark that for any $n \ge 1$, the map:

$$Y \to \mathbb{R}, \quad y \mapsto \sup_{(x,x')\in D_n} |F(y,x) - F(y,x')|,$$

is measurable, since D_n is countable. Moreover $\sup_{(x,x')\in D_n} |F(y,x) - F(y,x')| \le 2g(y)$, $n \ge 1$, $y \in Y$ $\mu-$ a.e.. Thus we get for any $n \ge 1$

$$\begin{aligned} \sup_{(x,x')\in A_n} |h(x) - h(x')| &\le \sup_{(x,x')\in A_n} \int_Y |F(y,x) - F(y,x')|\,\mu(dy) \\ &\le \int_Y \sup_{(x,x')\in D_n} |F(y,x) - F(y,x')|\,\mu(dy). \end{aligned} \tag{6.2.13}$$

Now letting $n \to \infty$ in the last term, by the Dominated Convergence Theorem, we find (6.2.12). ∎

Proposition 6.2.9 *Let* $\mathcal{A}$ *be the generator of a* π*-semigroup* P_t *of type* ω *on* $\mathcal{C}_b(E)$*, then we have:*
(i) $D(\mathcal{A})$ *is* π*- dense in* $\mathcal{C}_b(E)$*;*
(ii) $\mathcal{A}$ *is a* π*- closed operator on* $\mathcal{C}_b(E)$*.*

Proof Fix any $f \in \mathcal{C}_b(E)$ and consider for any $t > 0$ the following maps:

$$E \to \mathbb{R}, \quad x \mapsto \int_0^t P_s f(x) ds.$$

By Lemma 6.2.8 we already know that these maps belong to $\mathcal{C}_b(E)$. Let us prove that they belong to $D(\mathcal{A})$ for any $t > 0$. First remark that

$$P_h\Big(\int_0^t P_s f(\cdot)\, ds \Big)(x) \;=\; \int_0^t P_{h+s} f(x) ds, \; x \in E, \; t \geq 0, \tag{6.2.14}$$

since $\int_0^t P_s f(\cdot)\, ds$ is a π-limit of a sequence of Riemann sums in $\mathcal{C}_b(E)$. Indeed let us consider, for any $n \geq 1$, $(q_k^n) = (\frac{kt}{n})$, $k = 0, \ldots n$. We have, for any $x \in E$,

$$\lim_{n \to \infty} \frac{t}{n} \sum_{k=0}^{n-1} P_{q_k^n} f(x) = \int_0^t P_s f(x) ds. \tag{6.2.15}$$

Moreover it holds

$$\Big\| \frac{t}{n} \sum_{k=0}^{n-1} P_{q_k^n} f \Big\|_0 \leq \frac{t}{n} \sum_{k=0}^{n-1} \Big\| P_{q_k^n} f \Big\|_0 \leq M \|f\|_0 \frac{t}{n} \sum_{k=0}^{n-1} e^{\omega q_k^n} \;\leq\; M \|f\|_0 t e^{\omega t}, \;\; n \geq 1.$$

Hence, defining $f_n(x) = \dfrac{t}{n} \displaystyle\sum_{k=0}^{n-1} P_{q_k^n} f(x)$, $x \in E$, we obtain that $f_n \xrightarrow{\pi} \int_0^t P_s f(\cdot) ds$ as $n \to \infty$. Moreover one has, in a similar way, for any $h \geq 0$,

$$P_h f_n = \frac{t}{n} \sum_{k=0}^{n-1} P_{h + q_k^n} f \;\xrightarrow{\pi}\; \int_0^t P_{h+s} f(\cdot) ds, \;\; \text{as} \;\; n \to \infty$$

Now formula (6.2.14) follows, since $P_h f_n \xrightarrow{\pi} P_h(\int_0^t P_s f ds)$ as $n \to \infty$. Using (6.2.14), by simple computations, we infer, for any $t > 0$, $h > 0$, $x \in E$,

$$\begin{aligned} \Delta_h \int_0^t P_s f(x) ds &= \frac{1}{h}[\int_0^t P_{s+h} f(x) ds - \int_0^t P_s f(x) ds], \\ &= \frac{1}{h}[\int_h^{t+h} P_s f(x) ds \;-\; \int_0^t P_s f(x) ds] \\ &= \frac{1}{h}[\int_t^{t+h} P_s f(x) ds \;-\; \int_0^h P_s f(x) ds], \end{aligned} \tag{6.2.16}$$

Now as $h \to 0^+$, the right-hand side tends to $P_t f(x) - f(x)$, for any $x \in E$.

Remark that by (6.2.16), we find

$$\|\Delta_h \int_0^t P_s f ds\|_0 \;\leq\; 2M\, e^{\omega(t+1)}\, \|f\|_0, \;\; h \in]0, 1]$$

so that $\int_0^t P_s f(\cdot) ds \in D(\mathcal{A})$, $t \geq 0$. Now we can prove assertion (i) and (ii).

(i) Choose a sequence (t_n) of positive numbers, such that $t_n \to 0$, and consider

$$f_n(x) \;=\; \frac{1}{t_n} \int_0^{t_n} P_s f(x) ds, \;\; x \in E.$$

We have $(f_n) \subset D(\mathcal{A})$ and $f_n \xrightarrow{\pi} f$. Hence (i) is proved.
(ii) Let $(g_n) \subset D(\mathcal{A})$, such that $g_n \xrightarrow{\pi} g$ and $\mathcal{A} g_n \xrightarrow{\pi} \phi$.

Using the property (ii) of Proposition 6.2.7 we deduce

$$P_t g_n(x) - g_n(x) \;=\; \int_0^t P_s \mathcal{A} g_n(x) ds, \quad x \in E,\ t \geq 0,\ n \geq 1,$$

letting $n \to \infty$, since $P_s \mathcal{A} g_n \xrightarrow{\pi} P_s \phi$, $s \geq 0$, we find

$$P_t g(x) - g(x) \;=\; \int_0^t P_s \phi(x) ds, \quad x \in E,\ t \geq 0.$$

Dividing by $t > 0$ and letting $t \to 0^+$ we get that $\phi \in D(\mathcal{A})$ and $\mathcal{A} g = \phi$. ■

As an application of the previous result, we show that the maps which belong to $D(\mathcal{A})$ separate the points of E.

Corollary 6.2.10 *Let $\mathcal{A}$ be the generator of a π-semigroup P_t on $\mathcal{C}_b(E)$. Then for any z, $y \in E$, such that $y \neq z$, there exists a map $\psi \in D(\mathcal{A})$ such that $\psi(y) \neq \psi(z)$.*

Proof Take a map $f \in \mathcal{C}_b(E)$, such that $f(y) \neq f(z)$. For instance set

$$f(x) = \frac{d(x,y)}{d(x,y) + d(x,z)}, \quad x \in E.$$

By Proposition 6.2.9 we know that there exists a sequence $(f_n) \subset D(\mathcal{A})$ such that $f_n \xrightarrow{\pi} f$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty} f_n(y) = f(y), \quad \lim_{n\to\infty} f_n(z) = f(z).$$

It follows that there exists $n_0 \geq 1$, such that $f_{n_0}(y) \neq f_{n_0}(z)$. Thus $\psi = f_{n_0}$ is the desired map. ■

Let P_t be a π-semigroup on $\mathcal{C}_b(E)$ such that $\|P_t\|_{\mathcal{L}(\mathcal{C}_b(E))} \leq M\, e^{\alpha t}$, $t \geq 0$ with $\alpha \in \mathbb{R}$ and $M \geq 1$. Consider the following operators

$$F_\lambda f(x) \;=\; \int_0^\infty e^{-\lambda u}\, P_u f(x)\, du, \;\; f \in \mathcal{C}_b(E),\ x \in E,\ \lambda > \alpha. \tag{6.2.17}$$

By Lemma 6.2.8, we deduce that each $F_\lambda : \mathcal{C}_b(E) \to \mathcal{C}_b(E)$, $\lambda > \alpha$. Moreover $(F_\lambda)_{\lambda > \alpha}$ is a family of linear and continuous operators on $\mathcal{C}_b(E)$ and we have:

$$\|F_\lambda f\|_0 \leq M \|f\|_0 \int_0^\infty e^{[\alpha - \lambda] u}\, du \;=\; \frac{M}{\lambda - \alpha} \|f\|_0. \tag{6.2.18}$$

Let $\mathcal{A}$ be the generator of P_t, by Proposition 6.2.9 we know that in particular $\mathcal{A}$ is a closed operator. Next we characterize the resolvent operator $R(\lambda, \mathcal{A})$ of $\mathcal{A}$.

Proposition 6.2.11 *Let P_t be a π-semigroup with generator $\mathcal{A}$ such that $\|P_t\|_{\mathcal{L}} \leq M\, e^{\alpha t}$, $t \geq 0$ with $\alpha \in \mathbb{R}$ and $M \geq 1$. Consider the operators $(F_\lambda)_{\lambda > \alpha}$ defined in (6.2.17). Then it holds for any $\lambda > \alpha$:*

(i) there exists $R(\lambda, \mathcal{A}) = F_\lambda$;

(ii) we have $\|R(\lambda, \mathcal{A})^n\|_{\mathcal{L}(\mathcal{C}_b(E))} \leq \frac{M}{(\lambda - \alpha)^n}$, $n \geq 1$

Proof (i) First we prove that for $f \in \mathcal{C}_b(E)$, $\lambda > \alpha$, $F_\lambda f \in D(\mathcal{A})$ and moreover

$$(\lambda - \mathcal{A})F_\lambda f \;=\; f. \tag{6.2.19}$$

We fix $f \in \mathcal{C}_b(E)$ and $\lambda > \alpha$ and define the maps $g = F_\lambda f$ and g_T, $g_T(x) = \int_0^T e^{-\lambda u} P_u f(x)du$, $x \in E$, $T > 0$. We easily obtain

$$\lim_{T\to\infty} \|g_T - g\|_0 \le \lim_{T\to\infty} M\|f\|_0 \int_T^\infty e^{(\alpha-\lambda)u}\, du \;=\; 0. \tag{6.2.20}$$

Take into account that $P_h g_T(x) = \int_0^T e^{-\lambda u} P_{u+h} f(x)du$, $x \in E$, $T > 0$, $h \ge 0$, since g_T is a π-limit of Riemann sums in $\mathcal{C}_b(E)$, see (6.2.15). We have

$$\Delta_h g(x) \;=\; \Big(\frac{P_h - I}{h}\Big) g(x) \;=\; \frac{1}{h}\int_0^\infty e^{-\lambda u}\,[P_{u+h}f(x) - P_u f(x)]\,du$$

$$= \frac{1}{h}\Big(e^{\lambda h}\int_h^\infty e^{-\lambda v} P_v f(x)dv \;-\; \int_0^\infty e^{-\lambda u} P_u f(x)\,du\Big)$$

$$= \Gamma_1 f(x,h) \;-\; \Gamma_2 f(x,h) \quad \text{where} \quad \Gamma_1 f(x,h) = \frac{e^{\lambda h} - 1}{h}\int_0^\infty e^{-\lambda u} P_u f(x)du,$$

$$\Gamma_2 f(x,h) \;=\; \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda u}\, P_u f(x)du, \quad x \in E,\ h > 0. \tag{6.2.21}$$

Thus we have $\|\Delta_h g\|_0 \le M\|f\|_0\, e^\lambda[\frac{\lambda}{\lambda-\alpha} + e^\alpha]$, $h \in]0,1]$. Further

$$\lim_{h\to 0^+} \sup_{x\in E} |\Gamma_1 f(h,x) - \lambda g(x)| = 0. \tag{6.2.22}$$

Concerning the second term $\Gamma_2 f(h,x)$ we can compute for any $x \in E$:

$$\begin{aligned} &|\Gamma_2 f(h,x) - f(x)| \;\le\; |\Gamma_2 f(h,x) - \frac{1}{h}\int_0^h f(x)du| \\ &\le \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda u}|P_u f(x) - f(x)|du \;+\; \|f\|_0 \frac{e^{\lambda h}}{h}\int_0^h [e^{-\lambda u} - 1]du, \end{aligned} \tag{6.2.23}$$

that tends to 0 as $h \to 0^+$, since $\lim_{t\to 0^+} P_t f(x) = f(x),\ x \in E.$

Thus we have verified that

$$\Delta_h g \xrightarrow{\pi} \lambda g - f, \quad \text{as } \ h \to 0^+$$

and consequently $g \in D(\mathcal{A})$ and $\mathcal{A}g = \lambda g - f$. It follows that $(\lambda - \mathcal{A})F_\lambda f \;=\; f.$

Now assume that $l \in D(\mathcal{A})$, we claim that for $\lambda > \alpha$, $F_\lambda \mathcal{A} l \;=\; \mathcal{A}F_\lambda l.$

This fact and (6.2.19) will be imply that $F_\lambda(\lambda - \mathcal{A})l = l$. To this purpose we get

$$F_\lambda \mathcal{A} l(x) = \int_0^\infty e^{-\lambda u} P_u \mathcal{A} l(x)du \;=\; \int_0^\infty e^{-\lambda u}\mathcal{A} P_u l(x)du$$
$$= \mathcal{A}\int_0^\infty e^{-\lambda u}\, P_u l(x)du \;=\; \mathcal{A}F_\lambda l(x), \quad x \in E.$$

We have used formula (6.2.20) and the following two facts: $\mathcal{A}$ is a π-closed operator on $\mathcal{C}_b(E)$ and, for any $T>0$, $\int_0^T e^{-\lambda u} P_u l du$ is a π - limit of a sequence of Riemann sums in $\mathcal{C}_b(E)$, see (6.2.15).

Thus we have proved that there exists $R(\lambda, \mathcal{A})$, $\lambda > \alpha$ and

$$R(\lambda, \mathcal{A}) f(x) = \int_0^\infty e^{-\lambda u} P_u f(x)\, du, \quad f \in \mathcal{C}_b(E),\ x \in E. \tag{6.2.24}$$

(ii) From (6.2.24) in a standard way (differentiating with respect to λ and using induction, see for instance Pazy [61]) we can obtain, for any $f \in \mathcal{C}_b(E)$, $n \geq 1$, $\lambda > \alpha$,

$$R(\lambda, \mathcal{A})^n f(x) = \frac{1}{(n-1)!} \int_0^\infty u^{n-1} e^{-\lambda u} P_u f(x) du, \quad x \in E, \tag{6.2.25}$$

and now formula (ii) easily follows. ■

Let A be the generator of a π-semigroup P_t on $\mathcal{C}_b(E)$ and set $G = \overline{D(\mathcal{A})}$. By the previous result, applying the classical Hille-Yosida Theorem, we obtain that the part $\mathcal{A}_G$ of $\mathcal{A}$ in G (i.e. $D(\mathcal{A}_G) = \{f \in D(\mathcal{A})$ such that $\mathcal{A} f \in G\}$ and $\mathcal{A}_G f = \mathcal{A} f$, $f \in D(\mathcal{A}_G)$) is the generator of a $\mathcal{C}_0$-semigroup T_t on G (G is endowed with the sup norm). Clearly P_t is an extension of T_t to the whole of $\mathcal{C}_b(E)$, for any $t \geq 0$.

We can establish a connection between π-semigroups and *integrated semigroups*, see for instance Arendt [3], Hieber and Kellerman [43], Thieme [77].

Integrated semigroups satisfy an "integrated version" of the semigroup law. More precisely a once integrated semigroup S_t on a Banach space X is a strongly continuous family of linear and bounded operators on X, having the following properties

$$S_t S_r = \int_0^t (S_{u+r} - S_u) du, \quad t, r \geq 0 \text{ and } \quad S_0 = 0.$$

The theory of integrated semigroups generalizes that of $\mathcal{C}_0$-semigroups (any $\mathcal{C}_0$- semigroup is naturally associated with an integrated semigroup). It also allows to consider an abstract Cauchy problem with operators which do not verify the Hille-Yosida conditions.

From Proposition 6.2.11, invoking Theorem 4.1 in Arendt [3], we obtain

Proposition 6.2.12 *Let $\mathcal{A}$ be the generator of a π-semigroup P_t on $\mathcal{C}_b(E)$. Then $\mathcal{A}$ also generates a once integrated semigroup S_t on $\mathcal{C}_b(E)$. Moreover for any $f \in \overline{D(\mathcal{A})}$, we have $S_t f = \int_0^t P_r f dr$ (the integral has to be interpreted in the Bochner sense).*

Let $\mathcal{S}$ be a non trivial covering of E (see Definition 6.2.1) and P_t be a π-semigroup, we consider another linear operator $\mathcal{A}_\mathcal{S} : D(\mathcal{A}_\mathcal{S}) \subset \mathcal{C}_b(E) \to \mathcal{C}_b(E)$, defined as follows:

$$\begin{cases} D(\mathcal{A}_\mathcal{S}) = \{f \in D(\mathcal{A}) \text{ such that for any } S \in \mathcal{S},\ \lim\limits_{h \to 0^+} \sup\limits_{x \in S} |\Delta_h f(x) - \mathcal{A} f(x)| = 0\} \\ \mathcal{A}_\mathcal{S} f(x) \stackrel{\text{def}}{=} \mathcal{A} f(x), \quad f \in D(\mathcal{A}_\mathcal{S}),\ x \in E, \end{cases} \tag{6.2.26}$$

where $\Delta_h = h^{-1}(P_h - I)$. The proof of Proposition 6.2.11 can be suitably adapted in order to prove the following result.

Theorem 6.2.13 *Let P_t be a π-semigroup in $\mathcal{C}_b(E)$ of type ω and denote by $\mathcal{A}$ its generator. Let $\mathcal{S}$ be a non trivial covering of E such that formula (6.2.4) is verified by $\mathcal{S}$ and P_t. Then we have that* $\mathcal{A}_{\mathcal{S}} = \mathcal{A}$.

Proof Since $\mathcal{A}$ is an extension of $\mathcal{A}_{\mathcal{S}}$, we only have to prove that $D(\mathcal{A}) \subset D(\mathcal{A}_{\mathcal{S}})$.

To this end, first fix $g \in D(\mathcal{A})$ and $\lambda > \omega$. Then define $f = (\lambda - \mathcal{A})g$ so that

$$g(x) = \int_0^\infty e^{-\lambda u}\, P_u f(x)\, du, \quad x \in E.$$

We prove that $g \in D(\mathcal{A}_{\mathcal{S}})$ and $\mathcal{A}_{\mathcal{S}} g = \lambda g - f$.

Fix $S \in \mathcal{S}$ and take into account the proof of part (i) of Proposition 6.2.11. With the same notations we have $\Delta_h g(x) = \Gamma_1 f(h,x) - \Gamma_2 f(h,x), \;\; x \in E,\; h > 0$. It follows that

$$\begin{aligned} &\sup_{x \in S} |\Delta_h g(x) - \lambda g(x) + f(x)| \\ &\le \sup_{x \in S} |\Gamma_1 f(x,h) - \lambda g(x)| + \sup_{x \in S} |\Gamma_2 f(x,h) - f(x)|. \end{aligned} \tag{6.2.27}$$

Now by (6.2.22) we know that $\lim_{h \to 0^+} \sup_{x \in S} |\Gamma_1 f(x,h) - \lambda g(x)| = 0$. Let us consider the second term of (6.2.27).

$$\begin{aligned} &\sup_{x \in S} |\Gamma_2 f(x,h) - f(x)| \\ &\le \frac{e^{\lambda h}}{h} \sup_{x \in S} \int_0^h e^{-\lambda u} |P_u f(x) - f(x)| du + \|f\|_0 \frac{e^{\lambda h}}{h} \int_0^h [e^{-\lambda u} - 1] du, \end{aligned} \tag{6.2.28}$$

that tends to 0 as $h \to 0^+$, since by our hypotheses $\lim_{h \to 0^+} \sup_{x \in S} |P_h f(x) - f(x)| = 0$. The proof is complete. ∎

By the previous theorem, we derive the following useful result.

Corollary 6.2.14 *Let P_t be a π-semigroup on $\mathcal{C}_b(E)$ with generator $\mathcal{A}$. Suppose in addition that it is a strongly continuous semigroup on $\mathcal{C}_b(E)$. Denote by $\mathcal{A}_E$ its generator as a strongly continuous semigroup. Then we have* $\mathcal{A} = \mathcal{A}_E$

Proof We can apply the previous theorem with $\mathcal{S} = \{E\}$. ∎

Let us introduce the important class of transition π-semigroups. We also consider Dynkin's weak generator for transition functions that is similar to our generator of π-semigroups (we refer to Dynkin [31, ch. II §2] for more details).

Definition 6.2.15 A *(Markov) transition function* on a separable metric space (E,d), with the Borel σ- algebra denoted by $\mathcal{B}$, is $p(t,x,B) \ge 0$, where $t \ge 0$, $B \in \mathcal{B}$ and $x \in E$, that satisfies (denoting by δ_x the Dirac measure of $x \in E$):

(i) $p(t,x,\cdot)$ is a Borel measure on E such that $p(t,x,E) \le 1, \;\; t \ge 0,\; x \in E$;

(ii) $p(t,\cdot,B)$ is a real Borel map on $E, \;\; t \ge 0,\; B \in \mathcal{B}$;

(iii) $p(0,x,B) = \delta_x(B), \;\; x \in E, \;\; B \in \mathcal{B}$;

(iv) $p(t+s,x,B) = \int_E p(t,y,B)\, p(s,x,dy), \;\; s,\, t \ge 0,\; x \in E,\; B \in \mathcal{B}$.

Motivated by applications, see Section 6.3.2, we do not assume that $p(t,x,\cdot)$ is a probability measure. Any transition function $p(t,x,B)$ on E, defines a *(Markov) transition semigroup* T_t on $\mathcal{B}_b(E)$, the Banach space of all bounded, real and Borel functions on E (endowed with the sup norm)

$$T_t f(x) = \int_E f(y)\, p(t,x,dy), \quad f \in \mathcal{B}_b(E),\ x \in E,\ t \geq 0.$$

If the transition semigroup T_t on $\mathcal{B}_b(E)$ satisfies the additional conditions:

(v) for any $x \in E$, $f \in \mathcal{C}_b(E)$ the map $t \to T_t f(x)$ is continuous,

(vi) $T_t(\mathcal{C}_b(E)) \subset \mathcal{C}_b(E)$, $t \geq 0$,

then the restriction of T_t to $\mathcal{C}_b(E)$, that we denote again by T_t, is a π-semigroup. Note that hypothesis (iii) of Definition 6.2.3 follows immediately by the Dominated Convergence Theorem.

We call T_t a **transition π-semigroup** on $\mathcal{C}_b(E)$.

Now we deal with the Dynkin generator. Given a transition semigroup T_t on $\mathcal{B}_b(E)$, Dynkin introduces the space $\mathcal{B}_b^0(E) = \{f \in \mathcal{B}_b(E)$ such that $\lim_{t\to 0^+} T_t f(x) = f(x)$, $x \in E$ and there exists $\delta > 0$, $\|T_t f\|_0 \leq M$, for any $t \in [0,\delta]\,\}$.

Moreover he defines the *weak generator* $\tilde{\mathcal{A}}$ of T_t in the following way. $D(\tilde{\mathcal{A}}) = \{f \in \mathcal{B}_b(E)$ such that there exists $g \in \mathcal{B}_b^0(E)$: $\lim_{t\to 0^+} t^{-1}[T_t f(x) - f(x)] = g(x)$, $x \in E$ and there exists $\delta > 0$, $\|t^{-1}[T_t f - f]\|_0 \leq M$, for any $t \in]0,\delta]\}$. For any $f \in D(\tilde{\mathcal{A}})$,

$$\tilde{\mathcal{A}} f(x) \overset{\text{def}}{=} \lim_{t\to 0^+} t^{-1}[T_t f(x) - f(x)] \quad x \in E.$$

In Dynkin [31, ch. II] there are results about charaterization of transition functions by means of the weak generators of corresponding transition semigroups. ∎

The next result provides a characterization of transition π-semigroup. In the proof we use standard arguments.

Proposition 6.2.16 *Let P_t be a π-semigroup of contractions on $\mathcal{C}_b(E)$. P_t is a transition π-semigroup if and only if it is a positive semigroup on $\mathcal{C}_b(E)$ (i.e. for any $f \in \mathcal{C}_b(E)$, $f(x) \geq 0$, $x \in E$, implies that $P_t f(x) \geq 0$, $x \in E$, $t \geq 0$).*

Proof Clearly if P_t is a transition π-semigroup, it is positive. Let us prove the reverse implication and suppose that P_t is positive.

Consider the family of linear positive functionals $\{p_{t,x}\}_{t\geq 0,\, x\in E}$ on $\mathcal{C}_b(E)$,

$$f \mapsto p_{t,x}(f) \overset{\text{def}}{=} P_t f(x), \quad f \in \mathcal{C}_b(E),\ t \geq 0,\ x \in E.$$

For any $(f_n) \subset \mathcal{C}_b(E)$, such that $f_n \uparrow f$ as $n \to \infty$ (this means that $f_n(x) \leq f_{n+1}(x)$, $n \geq 1$, and $f_n(x)$ converges to $f(x)$, $x \in E$), with $f \in \mathcal{C}_b(E)$, we have that $P_t f_n \uparrow P_t f$ as $n \to \infty$ for any $t \geq 0$. Hence $p_{t,x}$ is an abstract integral on $\mathcal{C}_b(E)$, $t \geq 0$, $x \in E$. By the Daniell Theory, see for instance Ash [4], there exists for any $t \geq 0$, $x \in E$, a positive finite measure $p(t,x,\cdot)$ on the σ- algebra $\mathcal{T}$ generated by

$\mathcal{C}_b(E)$ (i.e. the smallest σ- algebra on E that makes each $f \in \mathcal{C}_b(E)$ measurable). Hence it holds:

$$p_{t,x}(f) \;=\; P_t f(x) \;=\; \int_E f(y)\, p(t,x,dy), \quad f \in \mathcal{C}_b(E),\; t \geq 0,\; x \in E. \tag{6.2.29}$$

We prove that $p(t,x,\cdot)$, $t \geq 0$, $x \in E$, are transition functions.

First we verify that $\mathcal{T} = \mathcal{B}$. Clearly $\mathcal{T} \subset \mathcal{B}$ so we establish the reverse inclusion. Denoting by $\mathcal{C}$ the family of all closed subsets of E, it is enough to check that $\mathcal{C} \subset \mathcal{T}$. Take a closed set $F \subset E$ and use $d(x,F) = \inf_{y\in F} d(x,y), \quad x \in E$.

Define the map $f(x) = \exp[-d(x,F)], \quad x \in E$. It is clear that $f \in \mathcal{C}_b(E)$ and $F = f^{-1}(1)$. Hence $F \in \mathcal{T}$ and the assertion is proved. Now the Borel measures $p(t,x,\cdot)$ verify conditions (i), (ii), (iii) of Definition 6.2.15.

It remains to prove condition (iv). By the semigroup law, we have for any $f \in \mathcal{C}_b(E)$, $t,\, s \geq 0$, $x \in E$,

$$\int_E \Big(\int_E f(y)\, p(s,z,dy) \Big)\, p(t,x,dz) \;=\; \int_E f(y)\, p(s+t,x,dy). \tag{6.2.30}$$

Denote by I_B the indicator of any $B \subset E$. Fix $F \in \mathcal{C}$, we prove that (6.2.30) holds for f replaced by I_C, so that condition (iv) will be satisfied by all closed sets in E.

We consider the maps $f_n(x) = \exp[-nd(x,F)], \quad x \in E,\ n \geq 1$. Now $f_n \downarrow I_F$, $\|f_n\|_0 \geq 1$ and $f_n \in \mathcal{C}_b(E)$, for any $n \leq 1$. Putting f_n in (6.2.30) and applying the Dominated Convergence Theorem we obtain the assertion.

Let $\mathcal{S} = \{B \in \mathcal{B}$ such that (6.2.30) holds for $I_B\}$. One can show that $\mathcal{S}$ is a Dynkin-system (i.e. $E \in \mathcal{S}$; for $A, B \in \mathcal{S}$ with $B \subset A$ we have: $A \backslash B \in \mathcal{S}$; for $(A_n) \subset \mathcal{S}$ with $A_n \uparrow A$, we have: $A \in \mathcal{S}$). To this purpose take into account that for any $A, B \in \mathcal{S}$ with $B \subset A$ we have $I_{A\backslash B} = I_A - I_B$ and further for any $(A_n) \subset \mathcal{S}$ and $A_n \uparrow A$, we have that $I_{A_n} \uparrow I_A$.

Since $\mathcal{C} \subset \mathcal{S}$ and $\mathcal{C}$ is closed under finite intersection, appealing to Ash [4, §4.1.2], we find that $\mathcal{S}$ coincides with the smallest $\sigma-$ algebra containing $\mathcal{C}$, namely $\mathcal{S} = \mathcal{B}$. The proof is complete. ∎

Remark 6.2.17 Here we show that any π-semigroup P_t on $\mathcal{C}_b(E)$ is a weakly integrable semigroup on $\mathcal{C}_b(E)$ in the Jefferies sense (see [44] and [45]). We verify the Jefferies initial assumptions (S1) and (S2).

Denote by $< \cdot, \cdot >$ the duality between $\mathcal{C}_b(E)$ and $\mathcal{C}_b(E)'$ (the topological dual of $\mathcal{C}_b(E)$)) and consider the following space

$$\begin{aligned}\Lambda(E) \;=\; \{\xi \in \mathcal{C}_b(E)^{'},\ \text{such that for any}\ \ (u_n) \subset \mathcal{C}_b(E),\ \ u_n \xrightarrow{\pi} u,\ \text{as}\ \ n \to \infty \\ \text{implies that} \quad \lim_{n\to\infty} < \xi, u_n > = < \xi, u > \}.\end{aligned} \tag{6.2.31}$$

In the proof of Remark 6.2.4, it is proved that $\Lambda(E) \neq \mathcal{C}_b(E)'$ even if $E = \mathbb{R}$. By Theorem 6.2.3, $\mathcal{M}(E) \subset \Lambda(E)$ and so $\Lambda(E)$ separates the points of $\mathcal{C}_b(E)$. Moreover $\Lambda(E)$ is an invariant subspace with respect to the dual semigroup $P_t' \ : \mathcal{C}_b(E)' \to \mathcal{C}_b(E)'$, $t \geq 0$.

Indeed consider $\xi \in \Lambda(E)$ and $(f_n) \subset \mathcal{C}_b(E)$ such that $f_n \xrightarrow{\pi} f$ as $n \to \infty$. Since $P_t f_n \xrightarrow{\pi} P_t f$, we have for any $t \geq 0$,

$$< f_n, P_t'\xi > = < P_t f_n, \xi > \to < P_t f, \xi > = < f, P_t'\xi > \text{ as } n \to \infty$$

and so $P_t'\xi \in \Lambda(E)$, $t \geq 0$. Thus hypothesis (S1) is satisfied.

Let us consider hypothesis (S2). First for any $\xi \in \Lambda(E)$, $f \in \mathcal{C}_b(E)$, we claim that the map: $\mathbb{R}_+ \to \mathbb{R}$, $t \mapsto < P_t f, \xi >$ is continuous. To see this fact, fix $t \geq 0$ and take any sequence $(t_n) \subset \mathbb{R}_+$ such that $t_n \to t$. Since $P_{t_n} f \xrightarrow{\pi} P_t f$ as $n \to \infty$ we obtain $\lim_{n\to\infty} < P_{t_n} f, \xi > = < P_t f, \xi >$.

Now fix $f \in \mathcal{C}_b(E)$ and $\lambda > \omega$, we set $g = R(\lambda, \mathcal{A})f$ (where $\mathcal{A}$ is the generator of P_t and ω its type), then it holds:

$$< g, \xi > = \int_0^\infty e^{-\lambda u} < P_u f, \xi > du, \ \ \xi \in \Lambda(E). \tag{6.2.32}$$

Indeed define for any $T > 0$ the map $g_T(x) = \int_0^T e^{-\lambda u} P_u f(x) du$, $x \in E$. $g_T \in \mathcal{C}_b(E)$ and it is a π- limit of Riemann sums in $\mathcal{C}_b(E)$, see (6.2.15). Hence we obtain for any $T > 0$:

$$\int_0^T e^{-\lambda u} < P_u f, \xi > du = < \int_0^T e^{-\lambda u} P_u f du, \xi > = < g_T, \xi > .$$

Now letting $T \to \infty$, we get (6.2.32), since $g_T \to g$ in $\mathcal{C}_b(E)$ as $T \to \infty$. Thus formula (6.2.32) holds and (S2) is verified.

Hence we can say, using the Jefferies terminology, that P_t is a $\Lambda(E)$-semigroup on $\mathcal{C}_b(E)$. ∎

6.3 Examples of π-semigroups

This section is devoted to describing some basic transition π-semigroups (see Definition 6.2.15) that occur in PDE's with infinitely many variables.

One can show that many Markov transition semigroups associated with equations (0.1.1) and (0.1.2) are actually π-semigroups. Some results in this direction are contained in Zambotti [87]. Here we only consider some important cases: the heat semigroup, the semigroup associated with a infinite dimensional Dirchlet problem, see Chapter 5, and the Ornstein-Uhlenbeck semigroup.

Previous results will be applied to give a detailed description of their generators.

We recall some notations, see Chapter 1 for more details. We denote by H, a real separable Hilbert space with inner product $< \cdot, \cdot >$ and norm $|\cdot|$. Q will be a positive (i.e. non negative and one to one) self-adjoint trace class (or nuclear) operator on H (Tr (Q) will denote the trace of Q). We define by $\mathcal{L}_1(H)$ the subspace of $\mathcal{L}(H)$ of all trace class operators. $\mathcal{L}_1(H)$, endowed with the norm $\|T\|_1 = \text{Tr}\,(\sqrt{T^*T})$, $T \in \mathcal{L}_1(H)$, is a Banach space.

Fix once and for all an orthonormal basis of H, $\{e_k\}_{k\geq 1}$, that diagonalizes Q, for any $x \in H$, $Qx = \sum_{k=1}^\infty \lambda_k x_k e_k$ with $x_k = < x, e_k >$, $k \geq 1$.

We also consider the Gaussian measure $\mathcal{N}(x, tQ)$ on H, with mean $x \in H$ and covariance operator tQ, $t > 0$.

6.3.1 The Heat semigroup

We recall the definition of the heat semigroup O_t on $\mathcal{C}_b(H)$, associated with the operator Q:

$$O_t f(x) \;=\; \int_H f(x+\sqrt{t}y)\,\mathcal{N}(0,Q)dy,\;\; f \in \mathcal{C}_b(H),\; x \in H,\; t \geq 0. \tag{6.3.1}$$

It is well known that it is a strongly continuous semigroup on $\mathcal{C}_b(H)$; we denote by $\mathcal{A}_H$ its generator. O_t is clearly also a transition π-semigroup and so, denoting by $\mathcal{A}$, its generator, by Corollary 6.2.14, we obtain that $\mathcal{A} \,=\, \mathcal{A}_H$.

6.3.2 A semigroup associated with a Dirichlet problem in a half space of H

We are dealing with the transition semigroup P_t considered in Chapter 5 (see also Priola [66]). We define an open half space of H with respect to the orthonormal basis, $\{e_k\}_{k\geq 1}$, previously fixed. Notice that each element x of H will be identified with its coordinates with respect to this basis.

$$H_+ \stackrel{\text{def}}{=} \{\; x = (x_1, x') \in H \;\text{ such that }\; x_1 > 0 \;\},$$

Let H' be the Hilbert subspace generated by $\{e_k\}_{k\geq 2}$. We set $Q'x' = \sum_{k=2}^{\infty} \lambda_k x'_k e_k$, $x' = (x'_k) \in H'$.

Then it holds: $H_+ \,=\, \mathbb{R}_+ \times H'$, where $\mathbb{R}_+ \,=\, (0, \infty)$.

Now we construct a semigroup P_t, associated with the following infinite dimensional Dirichlet problem

$$\begin{cases} \lambda\psi(x) \;-\; \dfrac{1}{2}\mathrm{Tr}\,[QD^2\psi(x)] = f(x), & x \in H_+,\;\; \lambda > 0, \\ \psi(z) = 0, \quad\quad z \in \partial H_+, & \end{cases} \tag{6.3.2}$$

where $f \in \mathcal{C}_b(H_+)$ and ∂H_+ denotes the boundary of H_+.

For any $g \in \mathcal{C}_b(H_+)$, we set $Eg(x) \,=\, g(x)$ if $x = (x_1, x')$ with $x_1 \geq 0$, $Eg(x) \,=\, -g(-x_1, x))$ if $x = (x_1, x')$ with $x_1 < 0$. Now we define the semigroup P_t, see also (5.2.3). For any $f \in \mathcal{C}_b(H_+)$, $t > 0$, $x \in H_+$,

$$\begin{aligned} P_t f(x) \;&=\; \int_H Ef(x+\sqrt{t}\,y)\,\mathcal{N}(0,Q)dy \\ &= \int_{\mathbb{R}_+\times H'} f(y_1, y')\; D(x_1, t\lambda_1) \otimes \mathcal{N}(x', tQ')(dy_1, dy') \end{aligned}$$

$$\text{where }\; D(x_1, t\lambda_1)(dy_1) \;=\; \left(\frac{e^{-\frac{(x_1-y_1)^2}{2t\lambda_1}} - e^{-\frac{(x_1+y_1)^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}} \right) dy_1,\;\; \lambda_1 > 0,\; t > 0,\; x_1 > 0 \tag{6.3.3}$$

and $\mathcal{N}(x', tQ')$ is a Gaussian measure on H'. Notice that $D(x_1, t\lambda_1) \otimes \mathcal{N}(x', tQ')(H_+) < 1$, $x \in H_+$, $t > 0$.

It is possible to verify that P_t is a semigroup of contractions on $\mathcal{C}_b(H_+)$. Clearly P_t is a transition π-semigroup on $\mathcal{C}_b(H_+)$, see Proposition 5.2.3, but is not a strongly continuous semigroup. Indeed in Proposition 5.2.6 it is verified that the *maximal subspace* on which P_t is a strongly continuous semigroup is
$\mathcal{C}_0(H_+) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_b(H_+)$ such that $f(z) = 0 \quad \forall z \in \partial H_+\}$.

Fix $t > 0$, then for any $f \in \mathcal{C}_b(H_+)$ it is possible to prove that $P_t f \in \mathcal{C}_0(H_+)$. This fact implies that P_t is not a π-semigroup on $\mathcal{C}_b(\overline{H_+})$ (compare with (b) of Remark 6.2.5).

Let $\mathcal{T}$ be the generator of P_t . Notice that $\mathcal{T}$ coincides (by Proposition 6.2.11) with the generator introduced in Proposition 5.2.8 by the pointwise Laplace transform of P_t. We define the following subsets of H_+:

$$H_+^\eta \stackrel{\text{def}}{=} \{(x_1, x') \in H_+ \quad / \quad x_1 \geq \eta\}, \quad \eta > 0 \tag{6.3.4}$$

Proposition 6.3.1 *For any $f \in \mathcal{C}_b(H_+)$, it holds:*
(i) $\lim_{s \to 0^+} P_s f = f$ *uniformly on each H_+^η, for any $\eta > 0$;*
(ii) $\lim_{s \to 0} P_{s+t} f = P_t f$ *uniformly on H_+, for any $t > 0$.*

Proof. (i) Let us fix $\eta > 0$ and prove that

$$\lim_{s \to 0^+} \sup_{x \in H_+^\eta} |P_s f(x) - f(x)| = 0. \tag{6.3.5}$$

Thanks to the separability of H, we can choose a countable dense subset D^η of H_+^η. Since $P_s f - f \in \mathcal{C}_b(H_+)$ for any $s \geq 0$, formula (6.3.5) is equivalent to the following one: $\lim_{s \to 0^+} \sup_{x \in D^\eta} |P_s f(x) - f(x)| = 0$.

We introduce the following real functions on H

$$F_s(y) \stackrel{\text{def}}{=} \sup_{x \in D^\eta} |\, Ef(x + \sqrt{s}y) - Ef(x)\,|, \quad s \geq 0,\ y \in H. \tag{6.3.6}$$

It turns out that: $\|F_s\|_0 \leq 2\|f\|_0$ and F_s is a Borel function on H, $s \geq 0$.

Furthermore we claim that

$$\lim_{s \to 0^+} F_s(y) = 0, \quad y \in H. \tag{6.3.7}$$

Indeed, by the uniform continuity of f, for any $\epsilon > 0$, there exists $\delta > 0$ such that
$|f(u) - f(v)| \leq \epsilon$ if $|u - v| \leq \delta$, $u, v \in H_+$.
Now for any fixed $\hat{y} = (\hat{y}_k)_{k \geq 1} \in H$, choose $s_0 > 0$ such that $|\sqrt{s_0}\hat{y}_1| \leq |\sqrt{s_0}\hat{y}| \leq \eta$ and $|\sqrt{s_0}\hat{y}| \leq \delta$.
We get $F_s(\hat{y}) \leq \epsilon$ for any $0 < s \leq s_0$ and so (6.3.7) follows. Finally consider the inequality

$$\sup_{x \in D^\eta} |P_s f(x) - f(x)| \leq \int_H F_s(y) \mathcal{N}(0, Q) dy.$$

Letting $s \to 0^+$ in the right-hand side, by the Dominated Convergence Theorem, we obtain (6.3.5) and so (i) is proved.

(ii) Fix $t > 0$, $f \in \mathcal{C}_b(H_+)$. Since $P_t f \in \mathcal{C}_0(H_+)$, we have $\lim_{r\to 0^+} P_r P_t f = P_t f$ in $\mathcal{C}_b(H_+)$. Hence to verify the assertion it remains to check that

$$\lim_{s\to 0^-} P_s P_t f = P_t f \quad \text{in} \quad \mathcal{C}_b(H_+).$$

To this purpose we have, for any $-t/2 \le s \le 0$,

$$\|P_t f - P_{s+t} f\|_0 = \|P_{s+t/2}(P_{t/2-s} f - P_{t/2} f)\|_0 \le \|P_{t/2-s} f - P_{t/2} f\|_0.$$

Taking into account that $P_{t/2} \in \mathcal{C}_0(H_+)$, we find

$$\lim_{s\to 0^-} \|P_t f - P_{s+t} f\|_0 \le \lim_{h\to 0^+} \|P_{t/2+h} f - P_{t/2} f\|_0 = 0.$$

The proof of (ii) is complete. ∎

Let us introduce the family $\mathcal{P} = \{H_+^\eta\}_{\eta>0}$. Similarly to (6.2.26), we can define the linear operator $\mathcal{T}_{\mathcal{P}} : D(\mathcal{T}_{\mathcal{P}}) \subset \mathcal{C}_b(H_+) \to \mathcal{C}_b(H_+)$, as follows:

$$\begin{cases} D(\mathcal{T}_{\mathcal{P}}) = \{ f \in D(\mathcal{T}) \text{ such that for any } H_+^\eta \in \mathcal{P}, \\ \displaystyle\lim_{h\to 0^+} \sup_{x\in H_+^\eta} \left|\frac{P_h f(x) - f(x)}{h} - \mathcal{T} f(x)\right| = 0 \}. \\ \\ \mathcal{T}_{\mathcal{P}} f(x) \stackrel{\text{def}}{=} \mathcal{T} f(x), \quad f \in D(\mathcal{T}_{\mathcal{P}}),\ x \in H_+. \end{cases} \tag{6.3.8}$$

By the previous proposition and by Theorem 6.2.13 we deduce

Proposition 6.3.2 *Let $\mathcal{T}$ be the generator of P_t, see (6.2.10). Then it holds: $\mathcal{T}_{\mathcal{P}} = \mathcal{T}$.*

6.3.3 The Ornstein-Uhlenbeck semigroup

Let A be the generator of a strongly continuous semigroup S_t on H, and let M be a self-adjoint and non negative bounded linear operator on H. For all $t \ge 0$, we define the bounded linear operators

$$M(t)x = \int_0^t S_u M S_u^* x \, du, \quad x \in H,$$

where S_t^* is the adjoint semigroup of S_t. Suppose that for each $t > 0$, $M(t)$ is a trace class operator. Under this assumption, there exist the Gaussian measures $\mathcal{N}(S_t x, M(t))$, $t > 0$, $x \in H$.

The Ornstein-Uhlenbeck semigroup on $\mathcal{C}_b(H)$ associated with S_t and M is defined as follows,

$$U_t f(x) = \int_H f(S_t x + y)\, \mathcal{N}(0, M(t))\, dy, \quad f \in \mathcal{C}_b(H),\ x \in H,\ t > 0. \tag{6.3.9}$$

This semigroup has been intensively studied, under various assumptions (see for instance Cerrai [14], Cerrai and Gozzi [15], Da Prato and Lunardi [21], Da Prato and

Zabczyk [23], Priola [69], Zambotti [86]).
Unless $S_t = I$, for any $t \geq 0$, U_t is not a strongly continuous semigroup on $\mathcal{C}_b(H)$ (see Cerrai [14, §6.1]). It can be verified that the maximal subspace of $\mathcal{C}_b(H)$ on which U_t is strongly continuous is

$$\mathcal{C}_b^S(H) = \{f \in \mathcal{C}_b(H) \text{ such that there exists } \lim_{t\to 0^+} \sup_{x\in H} |f(S_t x) - f(x)| = 0\}.$$

U_t turns out to be a transition π-semigroup on $\mathcal{C}_b(H)$. To this purpose it is enough to verify that the map $t \mapsto U_t f(x)$ is continuous for any $f \in \mathcal{C}_b(H)$, $x \in H$. Actually a stronger assertion holds. It was proved in Cerrai [14, §6.2, §6.3] in case S_t is a semigroup of contractions (let us notice that this hypothesis can be removed, with few changes in Cerrai's proof). Here we give a different and self-contained proof of this result.

Proposition 6.3.3 *For any compact set K in H, $f \in \mathcal{C}_b(H)$ one has*

$$\lim_{h\to 0} \sup_{x\in K} |U_{t+h} f(x) - U_t f(x)| = 0, \quad t \geq 0. \tag{6.3.10}$$

Proof Fix $t \geq 0$, $f \in \mathcal{C}_b(H)$ and a compact set K in H. Arguing by contradiction, if (6.3.10) fails to hold for t, f and K, there exist $\epsilon_0 > 0$ and two sequences $(t_n) \subset [0, \infty[$, with $t_n \to t$ as $n \to \infty$ and $(x_n) \subset K$ such that

$$|U_{t_n} f(x_n) - U_t f(x_n)| > \epsilon_0, \ \ n \geq 1.$$

There exists a subsequence (x_{n_j}) of (x_n) such that x_{n_j} converges to $z \in K$. Setting $n_j = j$ for convenience, we can write:

$$\epsilon_0 < |U_{t_j} f(x_j) - U_t f(x_j)| \leq \Psi^1(j) + \Psi^2(j) + \Psi^3(j), \ \ j \geq 1,$$

where

$$\Psi^1(j) = |U_{t_j} f(x_j) - U_{t_j} f(z)|, \ \ \Psi^2(j) = |U_{t_j} f(z) - U_t f(z)|$$
$$\Psi^3(j) = |U_t f(z) - U_t f(x_j)|.$$

Now we obtain a contradiction by proving that $\lim_{j\to\infty} \Psi^1(j) + \Psi^2(j) + \Psi^3(j) = 0$. First let us consider $\Psi^3(j)$. By the Dominated Convergence Theorem, it is clear that $\lim_{j\to\infty} \Psi^3(j) = 0$. As concerns $\Psi^2(j)$, denoting by ω_f the modulus of continuity of f, one has

$$\Psi^2(j) \leq \int_H |f(S_{t_j} z + y) - f(S_t z + y)| \mathcal{N}(0, M(t_j)) dy$$

$$+ \Big| \int_H f(S_t z + y) \mathcal{N}(0, M(t_j)) dy - \int_H f(S_t z + y) \mathcal{N}(0, M(t)) dy \Big|$$

$$\leq \omega_f(|S_{t_j} z - S_t z|) +$$

$$+ \Big| \int_H f(S_t z + y) \mathcal{N}(0, M(t_j)) dy - \int_H f(S_t z + y) \mathcal{N}(0, M(t)) dy \Big|.$$

Now consider that $\mathcal{N}(0, M(t_j))$ converges weakly to $\mathcal{N}(0, M(t))$ as $j \to \infty$, since $\|M(t_j) - M(t)\|_{\mathcal{L}_1(H)}$ tends to 0 as $j \to \infty$, see Proposition 1.1.5. Hence on making $j \to \infty$ in the last term of the previous formula, we get $\lim_{j\to\infty} \Psi^2(j) = 0$.

It remains to check $\Psi^1(j)$. We obtain for j large enough,

$$\Psi^1(j) \leq \int_H |f(S_{t_j}x_j + y) - f(S_{t_j}z + y)|\,\mathcal{N}(0, M(t_j))dy$$

$$\leq \omega_f(|S_{t_j}x_j - S_{t_j}z|) \leq \omega_f(Me^{\omega(t+1)}\,|x_j - z|),$$

that tends to 0 as $j \to \infty$. It follows that $\lim_{j\to\infty} \Psi^1(j) = 0$. This completes the proof. ∎

Denote by $\mathcal{U}$, the generator of the π-semigroup U_t. By Proposition 6.2.11, $\mathcal{U}$ coincides with the generator introduced in Cerrai [14] by using the pointwise Laplace transform of U_t.

Now we introduce, as in (6.2.26, the linear operator $\mathcal{U}_\mathcal{K} : D(\mathcal{U}_\mathcal{K}) \subset \mathcal{C}_b(H) \to \mathcal{C}_b(H)$, defined as follows:

$$\begin{cases} D(\mathcal{U}_\mathcal{K}) = \{f \in D(\mathcal{U})\,\text{such that for any } K \in \mathcal{K}, \\ \lim\limits_{h\to 0^+} \sup\limits_{x\in K} |\dfrac{U_h f(x) - f(x)}{h} - \mathcal{U}f(x)| = 0\,\}. \\ \\ \mathcal{U}_\mathcal{K} f(x) \overset{\text{def}}{=} \mathcal{U}f(x), \quad f \in D(\mathcal{U}_\mathcal{K}),\ x \in H. \end{cases} \tag{6.3.11}$$

By (6.3.10) and by Theorem 6.2.13 we deduce

Proposition 6.3.4 *Let $\mathcal{U}$ be the generator of U_t, see (6.2.10). Then it holds: $\mathcal{U}_\mathcal{K} = \mathcal{U}$.*

Finally we mention that there exist Markov transition semigroups on $\mathcal{C}_b(H)$ associated with non Gaussian transition functions, which verify condition (6.3.10). Among these semigroups there are the *Mehler semigroups*, studied in Fuhrman and Röckner [36], where also (6.3.10) is proved. Thus also for the Mehler semigroups, as for the Ornstein-Uhlenbeck semigroups, we can define a generator in three different equivalent ways: by a pointwise Laplace transform (as it is done in Fuhrman and Röckner [36, §4]), by a pointwise limit of an incremental ratio of the semigroup, see (6.2.10), and also by a uniform limit on compact sets of H of the same incremental ratio (see (6.3.11)).

6.4 A Hille-Yosida theorem for π-semigroups

In this section we give necessary and sufficient conditions on a given closed operator $\mathcal{T}$ on $\mathcal{C}_b(E)$ in order that it is the generator of a π-semigroup. We suppose that there exists an $\alpha \geq 0$ such that $]\alpha, \infty[$ belongs to the resolvent set of $\mathcal{T}$, which is denoted by $\rho(\mathcal{T})$.

In Cerrai [14] a generation theorem for weakly continuous semigroups is given, by using the Yosida approximations: $\mathcal{T}_n = n^2 R(n, \mathcal{T}) - nI$, $n > \alpha$. Here we consider the Hille approximations, defined as follows:

$$\mathcal{T}_{n,t} = (\frac{n}{t})^n R(\frac{n}{t}, \mathcal{T})^n = [(I - \frac{t}{n}\mathcal{T})^{-1}]^n, \quad n > \alpha t, \quad t \geq 0. \tag{6.4.1}$$

They seem to be simpler to handle in our case. Denoting by Λ_t an open neighborhood of $t \in [0, \infty[$, we state the main theorem.

Theorem 6.4.1 *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{C}_b(E) \to \mathcal{C}_b(E)$ be a linear operator. $\mathcal{A}$ is the generator of a π-semigroup P_t of type $\omega \geq 0$ if and only if the following statements hold:*
(i) $\mathcal{A}$ is closed;

(ii) $]\omega, \infty[\subset \rho(\mathcal{A})$ and there exists $M \geq 1$, such that
$\|R(\lambda, \mathcal{A})^n\|_{\mathcal{L}(\mathcal{C}_b(E))} \leq M\,(\lambda - \omega)^{-n}$ *for any $\lambda > \omega$, $n \geq 1$;*

(iii) for any $f \in \mathcal{C}_b(E)$,

$$\lim_{n \to \infty\, t \to 0^+} \mathcal{A}_{n,t} f(x) = f(x), \quad x \in E;$$

(iv) let $f \in \mathcal{C}_b(E)$, then for any $\epsilon > 0$, $t > 0$, there exists $\delta = \delta(\epsilon, t, f, \omega)$ such that for any $x, z \in E$, with $|x - z| \leq \delta$, there exists $\hat{n} = \hat{n}(x, z, \epsilon, t)$ for which

$$|\,\mathcal{A}_{n,t} f(x) - \mathcal{A}_{n,t} f(z)\,| \leq \epsilon, \quad n \geq \hat{n};$$

(v) let $(f_j) \subset \mathcal{C}_b(E)$ such that $f_j \xrightarrow{\pi} f$. For any $x \in E$, $\epsilon > 0$, $t_0 > 0$, there exists $j_0 = j_0(t_0, \epsilon, x, f, \omega)$ such that for any $j \geq j_0$ there exists Λ_{t_0} (depending on ϵ, j, x) and $n_0 = n_0(j, x, \epsilon, t_0)$ such that

$$|\,\mathcal{A}_{n,t} f_j(x) - \mathcal{A}_{n,t} f(x)\,| \leq \epsilon, \quad t \in \Lambda_{t_0}, \; n \geq n_0.$$

The proof of the necessity of (i), .. (v) will be carried out into some lemmas.

Lemma 6.4.2 *Let $\mathcal{A}$ be the generator of a π-semigroup P_t of type ω. Let $f \in \mathcal{C}_b(E)$, then for any $\epsilon > 0$, $t > 0$, there exists $\delta = \delta(\epsilon, t, f, \omega)$ such that for any $x, z \in E$, $|x - z| \leq \delta$, there exists $\hat{n} = \hat{n}(x, z, \epsilon)$ for which*

$$|\,\mathcal{A}_{n,t} f(x) - \mathcal{A}_{n,t} f(z)\,| \leq 3\,\epsilon, \quad n \geq \hat{n}$$

Proof We will work out all details of the present proof, whereas in the analogous proofs of the next propositions we will omit similar computations. Moreover we do not indicate the dependence of constants from f and ω for short.

First remark that, by formula (6.2.25), changing variable, we have that for any $f \in \mathcal{C}_b(E)$,

$$\begin{aligned} \mathcal{A}_{n,t}f(x) &= \frac{n^n}{t^n}\frac{1}{(n-1)!}\int_0^\infty v^{n-1}\, e^{-\frac{n}{t}v} P_v f(x)dv \\ &= \frac{n^n}{(n-1)!}\int_0^\infty u^{n-1}\, e^{-nu}\, P_{ut}f(x)du, \quad x \in E,\ t > 0,\ n > \omega t. \end{aligned} \tag{6.4.2}$$

Fix $t > 0,\ \epsilon > 0$. By the uniform continuity of $P_t f$ there exists $\delta = \delta(t, \epsilon) > 0$ such that for $x,\ y \in E,\ \ |x - y| \leq \delta$ implies that $|P_t f(x) - P_t f(z)| \leq \frac{\epsilon}{2}$.
Fix $x,\ z \in E$ such that $|x - z| \leq \delta$. By the continuity of $|P_{(\cdot)}f(z) - P_{(\cdot)}f(x)|$ from $[0, \infty]$ into $\mathbb{R}$, we can find an open neighboorhood Λ_t of t such that:

$$|P_s f(z) - P_s f(x)| \leq \epsilon, \quad s \in \Lambda_t. \tag{6.4.3}$$

Let us choose $0 < a < 1 < b$, where $a = a(x, z, t, \epsilon)$ and $b = b(x, z, t, \epsilon)$, such that $[at, bt] \subset \Lambda_t$.

By virtue of (6.4.2) we have:

$$\begin{aligned} &|\mathcal{A}_{n,t}f(x) - \mathcal{A}_{n,t}f(z)| \\ &\leq \frac{n^n}{(n-1)!}\int_0^\infty u^{n-1}\, e^{-nu}\, |P_{ut}f(x) - P_{ut}f(z)|du, \quad n > \omega t. \end{aligned} \tag{6.4.4}$$

We break the last integral into three parts, similarly to the proof of Pazy [61, §1.8.3],

$$\frac{n^n}{(n-1)!}\int_0^\infty u^{n-1}\, e^{-nu}|P_{ut}f(x) - P_{ut}f(z)|du = \Gamma_1 + \Gamma_2 + \Gamma_3,$$

$$\begin{aligned} \text{where } \Gamma_1 &= \frac{n^n}{(n-1)!}\int_0^a u^{n-1}\, e^{-nu}\, |P_{ut}f(x) - P_{ut}f(z)|\, du, \\ \Gamma_2 &= \frac{n^n}{(n-1)!}\int_b^\infty u^{n-1}\, e^{-nu}\, |P_{ut}f(x) - P_{ut}f(z)|du, \\ \Gamma_3 &= \frac{n^n}{(n-1)!}\int_a^b u^{n-1}\, e^{-nu}|P_{ut}f(x) - P_{ut}f(z)|du. \end{aligned} \tag{6.4.5}$$

Now we consider each term separately. As concerns Γ_1,

$$\Gamma_1 \leq 2M\, e^{\omega t}\|f\|_0\, \frac{n^n}{(n-1)!}\int_0^a u^{n-1}\, e^{-nu}\, du, \tag{6.4.6}$$

let us notice that for all n such that $na < n-1$, we have $u^{n-1}e^{-nu} \leq a^{n-1}e^{-na}$, $u \geq 0$. Hence there exists $n_0 = n_0(x, z, \epsilon, t) \geq 1$ such that:

$$\Gamma_1 \leq 2Ma\, e^{\omega t}\|f\|_0\, \frac{n^n}{(n-1)!}a^{n-1}\, e^{-na} \leq \epsilon, \quad n \geq n_0,$$

Let us estimate the term Γ_2, for any $n > \omega t$,

$$\begin{aligned}
\Gamma_2 &\le 2M\|f\|_0 \frac{n^n}{(n-1)!}\int_b^\infty u^{n-1}\, e^{-nu}\, e^{\omega t u} du \\
&\le C\,\frac{n^n}{(n-1)!}\int_b^\infty u^{n-1}\, e^{-u(n-1-\omega t)} e^{-u} du \\
&\le C\,\frac{n^n}{(n-1)!}\int_b^\infty [ue^{-u}]^{n-1-\omega t} e^{-u} u^{\omega t}\, du \\
&\le C\frac{n^n}{(n-1)!}[be^{-b}]^{n-1}\, b^{-\omega t}\, e^{b\omega t}\int_b^\infty e^{-u} u^{\omega t}\, du.
\end{aligned} \tag{6.4.7}$$

Thus there exists $n_1 = n_1(x, z, \epsilon, t)$ such that $\Gamma_2 \le \epsilon$, $n \ge n_1$.

It remains to consider Γ_3. Formula (6.4.3) implies that

$$\Gamma_3 \;\le\; \frac{n^n}{(n-1)!}\int_a^b u^{n-1}\, e^{-nu} du \;\le\; \epsilon, \quad n \ge 1, \tag{6.4.8}$$

taking into account that

$$\frac{n^n}{(n-1)!}\int_0^\infty u^{n-1}\, e^{-nu} du \;=\; 1, \;\; n \ge 1.$$

Now setting $\hat{n} = \max(n_0, n_1)$, we obtain our statement. ■

Lemma 6.4.3 *Let $\mathcal{A}$ be the generator of a π-semigroup P_t of type ω. Let $f \in \mathcal{C}_b(E)$, then we have*

$$\lim_{n\to\infty\, t\to 0^+} \mathcal{A}_{n,t} f(x) \;=\; f(x), \quad x \in E.$$

Proof Fix $x \in E$. We have to prove that for any $\epsilon > 0$, there exists$\delta = \delta(\epsilon, x) > 0$ and $n_0 = n_0(\epsilon, x)$ such that

$$|\mathcal{A}_{n,t} f(x) \,-\, f(x)| \;\le\; 2\epsilon, \;\; t \in [0, \delta], \;\; n \ge n_0. \tag{6.4.9}$$

We know that there exists $\delta = \delta(\epsilon, x) > 0$, such that $|P_s f(x) - f(x)| \;\le\; \epsilon$, $s \in [0, \delta]$. Thus we have

$$|P_{ut} f(x) \,-\, f(x)| \;\le \epsilon, \quad t \in [0, \frac{\delta}{2}], \;\; u \in [\frac{1}{2}, 2]. \tag{6.4.10}$$

Now arguing as in proof of Lemma 6.4.2, we consider for any $n > \omega t$

$$|\mathcal{A}_{n,t} f(x) \,-\, f(x)| \le \frac{n^n}{(n-1)!}\int_0^\infty u^{n-1}\, e^{-nu}|P_{ut} f(x) - f(x)|\, du$$

$$= \Gamma_1 \,+\, \Gamma_2 + \Gamma_3, \qquad \text{where}$$

$$\begin{aligned}
\Gamma_1 &= \frac{n^n}{(n-1)!}\int_0^{\frac{1}{2}} u^{n-1}\, e^{-nu}\, |P_{ut} f(x) - f(x)|\, du, \\
\Gamma_2 &= \frac{n^n}{(n-1)!}\int_2^\infty u^{n-1}\, e^{-nu}\, |P_{ut} f(x) - f(x)| du, \\
\Gamma_3 &= \frac{n^n}{(n-1)!}\int_{\frac{1}{2}}^2 u^{n-1}\, e^{-nu}|P_{ut} f(x) - f(x)| du.
\end{aligned} \tag{6.4.11}$$

Now we have that there exists $\hat{n} = \hat{n}(\epsilon, f, \omega) \geq 1$ such that

$$\Gamma_1 + \Gamma_2 \leq \epsilon, \quad n \geq \hat{n}$$

and further, using (6.4.10), for any $t \in [0, \frac{\delta}{2}]$, $n \geq 1$

$$\Gamma_3 \leq \epsilon \frac{n^n}{(n-1)!} \int_{\frac{1}{2}}^{2} u^{n-1} e^{-nu} du \leq \epsilon \tag{6.4.12}$$

Thus $|\mathcal{A}_{n,t} f(x) - f(x)| \leq 2\epsilon$, for any $t \in [0, \frac{\delta}{2}]$, $n \geq \hat{n}$. ∎

In a similar way to Lemma 6.4.3 one could prove the following version of a classical Hille's theorem, about strongly continuous semigroups (see for instance Pazy [61, §1.8.3]), adapted to π-semigroups. We will not use this result but state it for the sake of completeness.

Proposition 6.4.4 *Let $\mathcal{A}$ be the generator of a π-semigroup P_t of type ω. Let $f \in \mathcal{C}_b(E)$, then for any $x \in E$, $T > 0$ we have*

$$\lim_{n\to\infty} \mathcal{A}_{n,t} f(x) = P_t f(x), \quad \text{uniformly in} \quad t \in [0, T].$$

Lemma 6.4.5 *Let $\mathcal{A}$ be the generator of a π-semigroup P_t of type ω. Let $(f_n) \subset \mathcal{C}_b(E)$ such that $f_n \xrightarrow{\pi} f$.*

For any $x \in E$, $\epsilon > 0$, $t_0 > 0$, there exists $j_0 = j_0(t_0, \epsilon, x, f, \omega)$ such that for any $j \geq j_0$ there exists a neighboorhood V_{t_0} of t_0 (depending on ϵ, j, x) and $n_0 = n_0(j, x, \epsilon, t_0)$ such that

$$|\mathcal{A}_{n,t} f_j(x) - \mathcal{A}_{n,t} f(x)| \leq 2\epsilon, \quad t \in V_{t_0}, \ n \geq n_0.$$

Proof Fix $t_0 > 0$, $x \in E$, $\epsilon > 0$.
Since P_t preserves the π- convergence of sequences, there exists $j_0 = j_0(t_0, \epsilon, x) \geq 1$, such that

$$|P_{t_0} f_j(x) - P_{t_0} f(x)| \leq \frac{\epsilon}{2}, \quad j \geq j_0.$$

Let us remark that the map $|P_{(\cdot)} f_j(x) - P_{(\cdot)} f(x)|$ is continuous in t_0 for any $j \geq j_0$, so that there exists an open neighboorhood $\Lambda^j_{t_0}$ of t_0 (we choose $\Lambda^j_{t_0} \subset]\frac{t_0}{2}, \infty[$) for which

$$|P_t f_j(x) - P_t f(x)| \leq \epsilon, \quad t \in \Lambda^j_{t_0}. \tag{6.4.13}$$

Fix for any $j \geq j_0$, two constants a_j, b_j, with $0 < a_j < 1 < b_j$ such that $[a_j t_0, b_j t_0] \subset \Lambda^j_{t_0}$.

Now for any $j \geq j_0$, it is possible to choose another neighboorhood $V^j_{t_0}$ of t_0, $V^j_{t_0} \subset \Lambda^j_{t_0}$, such that for any $t \in V^j_{t_0}$, we have that $a_j t \in \Lambda^j_{t_0}$ and $b_j t \in \Lambda^j_{t_0}$. In this way we have obtained that

$$\text{for any} \quad t \in V^j_{t_0}, \quad v \in [a_j, b_j], \quad \text{we have} \quad tv \in \Lambda^j_{t_0}, \quad j \geq j_0. \tag{6.4.14}$$

Arguing as in proof of Lemma 6.4.2 and setting $C = \sup_{j\ge 1} \|f_j\|_0$, we obtain for any $n > \omega \frac{t_0}{2}$ and $t \in V^j_{t_0}$,

$$|\mathcal{A}_{n,t} f_j(x) - \mathcal{A}_{n,t} f(x)| \le \frac{n^n}{(n-1)!} \int_0^\infty u^{n-1} e^{-nu} |P_{ut} f_j(x) - P_{ut} f(x)|\, du$$

$$\le \Gamma_1 + \Gamma_2 + \Gamma_3, \qquad \text{where}$$

$$\Gamma_1 = M e^{\omega(t_0+1)} (\|f\|_0 + C) \frac{n^n}{(n-1)!} \int_0^{a_j} u^{n-1} e^{-nu}\, du, \tag{6.4.15}$$

$$\Gamma_2 = M(\|f\|_0 + C) \frac{n^n}{(n-1)!} \int_{b_j}^\infty u^{n-1} e^{-nu} e^{\omega t u} du,$$

$$\Gamma_3 = \frac{n^n}{(n-1)!} \int_{a_j}^{b_j} u^{n-1} e^{-nu} |P_{ut} f_j(x) - P_{ut} f(x)| du.$$

Now we know that there exists $n_0 = n_0(\epsilon, j, t_0)) \ge 1$ such that

$$\Gamma_1 + \Gamma_2 \le \epsilon, \quad n \ge n_0.$$

Further, using (6.4.13) and (6.4.14), for any $t \in V^j_{t_0}$, $n \ge 1$ we get

$$\Gamma_3 \le \epsilon \frac{n^n}{(n-1)!} \int_{a_j}^{b_j} u^{n-1} e^{-nu} du \le \epsilon \frac{n^n}{(n-1)!} \int_0^\infty u^{n-1} e^{-nu} du = \epsilon. \tag{6.4.16}$$

Thus for any $j \ge j_0$ it holds:

$$| \mathcal{A}_{n,t} f_j(x) - \mathcal{A}_{n,t} f(x) | \le 2\epsilon, \quad t \in V^j_{t_0},\ n \ge n_0. \ \blacksquare$$

Now we are ready to prove the main theorem of this chapter.

Proof of Theorem 6.4.1. *Necessity.* It follows collecting Propositions 6.2.9, 6.2.11 and Lemmas 6.4.2, 6.4.3, 6.4.5.

Sufficiency. We start from the classical Hille' s method to prove the Hille-Yosida Theorem for strongly continuous semigroups. We refer to Tanabe [75, §3.1.4] for the exposition of this method .

We set, by (6.4.1), $\mathcal{A}_{n,t} = (\frac{n}{t})^n R(\frac{n}{t}, \mathcal{A})^n$, $t \ge 0$, $n > \omega t$. Notice that by assumption (ii) we have the following estimate:

$$\limsup_{n\to\infty} \|\mathcal{A}_{n,t}\|_{\mathcal{L}} \le \lim_{n\to\infty} M \frac{n^n}{(n-\omega t)^n} = M e^{\omega t}, \quad t \ge 0. \tag{6.4.17}$$

Only using hypothesis (i), (ii), following Tanabe [75, §3.1.4], we obtain that there exists a family of linear operators: $e^{t\mathcal{A}} : D(\mathcal{A}^2) \to \mathcal{C}_b(E)$, $t \ge 0$, $e^{0\mathcal{A}} = I$, such that for any $f \in D(\mathcal{A}^2)$,

$$\begin{aligned} &(1)\ \lim_{n\to\infty} \mathcal{A}_{n,t} f \;=\; e^{t\mathcal{A}} f \text{ in } \mathcal{C}_b(E), \text{ uniformly on bounded sets of } [0,\infty[;\\ &(2)\ \|e^{t\mathcal{A}} f\|_0 \;\le\; M\, e^{\omega t}\|f\|_0;\\ &(3)\ \lim_{t\to t_0} e^{t\mathcal{A}} f \;=\; e^{t_0\mathcal{A}} f \text{ in } \mathcal{C}_b(E),\ t_0 \ge 0. \end{aligned} \tag{6.4.18}$$

In the sequel the proof is carried out into several steps.

Step 1. We prove that it holds for any t, $s \ge 0$:

$$e^{(t+s)\mathcal{A}} f \;=\; e^{t\mathcal{A}}\, e^{s\mathcal{A}} f, \quad f \in D(\mathcal{A}^5). \tag{6.4.19}$$

First we have $\frac{d}{dt}\mathcal{A}_{n,t} f = \frac{n}{t} R(\frac{n}{t})\mathcal{A}_{n,t}\mathcal{A} f$, if $f \in D(\mathcal{A})$, $n > \omega t$ and hence for $t > 0$

$$\mathcal{A}_{n,t} f - f \;=\; \int_0^t \tfrac{n}{s} R(\tfrac{n}{s})\, \mathcal{A}_{n,s}\mathcal{A} f\, ds, \;\; f \in D(\mathcal{A}), \tag{6.4.20}$$

where the integral is in the Riemann sense, $\mathcal{C}_b(E)$- valued. Further for any $f \in D(\mathcal{A}^3)$, we have:

$$\lim_{n\to\infty} \tfrac{n}{s} R(\tfrac{n}{s})\mathcal{A}_{n,s}\mathcal{A} f \;=\; e^{s\mathcal{A}}\mathcal{A} f, \quad s \ge 0. \tag{6.4.21}$$

To verify (6.4.21), we write for any $f \in D(\mathcal{A}^3)$, $s \ge 0$,

$$\left\|\tfrac{n}{s} R(\tfrac{n}{s})\mathcal{A}_{n,s}\mathcal{A} f \;-\; e^{s\mathcal{A}}\mathcal{A} f\right\|_0 \le \;\; \Gamma_1 \,+\, \Gamma_2, \;\; \text{where}$$

$$\Gamma_1 = \;\|\mathcal{A}_{n,s}\|_{\mathcal{L}} \left\|\tfrac{n}{s} R(\tfrac{n}{s})\, \mathcal{A} f - \mathcal{A} f\right\|_0, \quad \Gamma_2 = \;\left\|\mathcal{A}_{n,s}\mathcal{A} f - \; e^{s\mathcal{A}}\mathcal{A} f\right\|_0.$$

$\Gamma_2 \to 0$ as $n \to \infty$ by condition (1) of (6.4.18). As concerns Γ_1, using (6.4.17) we obtain for any $s \ge 0$ and n large enough:

$$\begin{aligned} \Gamma_1 \;&\le\; M\,(e^{\omega s} + 1) \left\|\tfrac{n}{s} R(\tfrac{n}{s})\, \mathcal{A} f - \mathcal{A} f\right\|_0 \\ &=\; M\,(e^{\omega s} + 1) \left\|\tfrac{n}{s} R(\tfrac{n}{s})\mathcal{A}^2 f\right\|_0 \;\le\; M^2\|\mathcal{A}^2 f\|_0 (e^{\omega s} + 1)\frac{s}{n - \omega s}. \end{aligned}$$

Thus we find that also $\lim_{n\to\infty}\Gamma_1 = 0$ and (6.4.21) is proved. Now taking into account estimates (6.4.17) we can apply the Dominated Convergence Theorem into (6.4.20). Hence letting $n \to \infty$ in (6.4.20), we obtain

$$\begin{aligned} &e^{t\mathcal{A}} f - f \;=\; \int_0^t \; e^{s\mathcal{A}}\mathcal{A} f\, ds \quad \text{and so}\\ &\frac{d}{dt} e^{t\mathcal{A}} \;=\; e^{t\mathcal{A}}\mathcal{A} f, \quad f \in D(\mathcal{A}^3), \;\; t \ge 0. \end{aligned} \tag{6.4.22}$$

Moreover we have:

$$e^{t\mathcal{A}} f \in D(\mathcal{A}^2) \;\text{ and }\; \mathcal{A}^2 e^{t\mathcal{A}} f = e^{t\mathcal{A}}\mathcal{A}^2 f \;\text{ for any }\; f \in D(\mathcal{A}^4),\; t \ge 0. \tag{6.4.23}$$

Indeed for fixed $f \in D(\mathcal{A}^4)$, $t \geq 0$, $e^{t\mathcal{A}} f = \lim_{n\to\infty} \mathcal{A}_{n,t} f$ and further $\lim_{n\to\infty} \mathcal{A}\mathcal{A}_{n,t} f = e^{t\mathcal{A}} \mathcal{A} f$.

Using the closedness of $\mathcal{A}$, we conclude that $e^{t\mathcal{A}} f \in D(\mathcal{A})$ and $\mathcal{A}(e^{t\mathcal{A}} f) = e^{t\mathcal{A}} \mathcal{A} f$ $t \geq 0$. Proceeding in the same way, we get formula (6.4.23).

From (6.4.23), it is meaningful to consider $e^{t\mathcal{A}} e^{s\mathcal{A}} f$, $f \in D(\mathcal{A}^4)$, $s, t \geq 0$. Thus one can derive that

$$\frac{d}{ds} \left(e^{(t-s)\mathcal{A}} \, e^{s\mathcal{A}} f \right) \; = \; 0, \;\; 0 < s < t, \;\; f \in D(\mathcal{A}^5). \tag{6.4.24}$$

To this end consider that $e^{t\mathcal{A}} f \in D(\mathcal{A}^3)$, for any $f \in D(\mathcal{A}^5)$ (arguing as in (6.4.23)) and then use (6.4.22).

Since $\lim_{s\to 0^+} e^{(t-s)\mathcal{A}} e^{s\mathcal{A}} f = e^{t\mathcal{A}} f$, for $f \in D(\mathcal{A}^4)$, $t > 0$, we deduce, by (6.4.24), that for any $t > 0$,

$$e^{(t-s)\mathcal{A}} e^{s\mathcal{A}} f = e^{t\mathcal{A}} f, \quad f \in D(\mathcal{A}^5), \;\; 0 \leq s \leq t,$$

and the property (6.4.19) is proved.

Step 2. Now we prove that for any $f \in \mathcal{C}_b(E)$ there exists the limit

$$\lim_{n\to\infty} \mathcal{A}_{n,t} f(x), \;\; x \in E, \; t \geq 0, \tag{6.4.25}$$

To this purpose we need to prove the following fact. For any $k \geq 1$ fixed, $D(\mathcal{A}^k)$ is π- dense in $\mathcal{C}_b(E)$.

Consider any $f \in \mathcal{C}_b(E)$, we prove that there exists a sequence $(f_j) \subset D(\mathcal{A}^k)$ such that $f_j \xrightarrow{\pi} f$ as $j \to \infty$. By assumption (iii), setting $t = \frac{1}{n}$, $n \geq 1$ we know that

$$\lim_{n\to\infty} \mathcal{A}_{n,\frac{1}{n}} f(x) \; = \; f(x), \;\; f \in \mathcal{C}_b(E), \;\; x \in E. \tag{6.4.26}$$

Moreover $\|\mathcal{A}_{n,\frac{1}{n}}\|_{\mathcal{L}} = \|n^{2n} R(n^2)^n\|_{\mathcal{L}} \leq M n^{2n} (n^2 - \omega)^{-n} \leq (M+1)$ for n large.

Thus $\mathcal{A}_{n,\frac{1}{n}} f \xrightarrow{\pi} f$ as $n \to \infty$. Setting $f_j = \mathcal{A}_{j,\frac{1}{j}} f$, $j \geq k$ we obtain the desired approximation for f.

To prove (6.4.25), fix $f \in \mathcal{C}_b(E)$, $t > 0$, $x \in E$ and choose a sequence $(f_j) \subset D(\mathcal{A}^2)$ such that $f_j \xrightarrow{\pi} f$ as $j \to \infty$. We have for any $n, \, m > \omega t, \; j \geq 1$,

$$\begin{aligned} |\mathcal{A}_{n,t} f(x) \, - \mathcal{A}_{m,t} f(x)| \; &\leq \; |\mathcal{A}_{n,t} f(x) - \mathcal{A}_{n,t} f_j(x)| \\ &+ |\mathcal{A}_{n,t} f_j(x) \, - \, \mathcal{A}_{m,t} f_j(x)| \, + \, |\mathcal{A}_{m,t} f_j(x) \, - \mathcal{A}_{m,t} f(x)|. \end{aligned} \tag{6.4.27}$$

By assumption (v) there exists $\hat{j} = \hat{j}(t, \epsilon, x)$ and $\hat{n} = \hat{n}(\hat{j})$ such that for amy $n \geq \hat{n}$ $|\mathcal{A}_{n,t} f(x) - \mathcal{A}_{n,t} f_{\hat{j}}(x)| \leq \epsilon$.

By (1) of (6.4.18) there exists $n_1 = n_1(\epsilon, \hat{j})$ such that

$$\|\mathcal{A}_{n,t} f_{\hat{j}} \, - \, \mathcal{A}_{m,t} f_{\hat{j}}\|_0 \; \leq \; \epsilon, \quad n, m \geq n_1.$$

Replacing f_j in (6.4.27) by $f_{\hat{j}}$ we find for any $n, m \geq \max(n_1, \hat{n})$: $|\mathcal{A}_{n,t} f(x) - \mathcal{A}_{m,t} f(x)| \leq 3\epsilon$.

The operator $e^{t\mathcal{A}}$ for any $t > 0$ can be extended to the whole of $\mathcal{C}_b(E)$, setting for any $f \in \mathcal{C}_b(E)$,

$$e^{t\mathcal{A}} f(x) \;=\; \lim_{n\to\infty} \mathcal{A}_{n,t} f(x), \;\; x \in E, \; t \geq 0. \tag{6.4.28}$$

Step 3. Let us show that $e^{t\mathcal{A}} f \in \mathcal{C}_b(E)$, for any $f \in \mathcal{C}_b(E)$ and $e^{t\mathcal{A}} \in \mathcal{L}(\mathcal{C}_b(E))$, $t \geq 0$.

First we verify the uniform continuity of $e^{t\mathcal{A}} f$. Fix $\epsilon > 0, \; t > 0$, by assumption (iv), we can choose $\delta = \delta(\epsilon, t) > 0$ such that for fixed $x, z \in E$ with $|x - z| \leq \delta$, there exists $n_0(x,z)$ for which it holds:

$$|\mathcal{A}_{n,t} f(x) \;-\; \mathcal{A}_{n,t} f(z)| \;\leq\; \epsilon, \quad n \geq n_0.$$

Consider the inequality:

$$|e^{t\mathcal{A}} f(x) - e^{t\mathcal{A}} f(z)| \;\leq\; |e^{t\mathcal{A}} f(x) - \mathcal{A}_{n,t} f(x)| + |\mathcal{A}_{n,t} f(x) - \mathcal{A}_{n,t} f(z)|$$

$$+ \, |\mathcal{A}_{n,t} f(z) - e^{t\mathcal{A}} f(z)| \;=\; L_1 \,+\, L_2 \,+ L_3$$

By (6.4.28), there exists $n_1 = n_1(x, z, t, \epsilon)$ such that $L_1 + L_2 \leq \epsilon$, $n \geq n_1$. Thus for any $n \geq \max(n_1, n_0)$:

$$|e^{t\mathcal{A}} f(x) - e^{t\mathcal{A}} f(z)| \;\leq\; 2\epsilon,$$

and the uniform continuity of $e^{t\mathcal{A}} f$, $f \in \mathcal{C}_b(E)$ is proved. Further we have

$$\begin{aligned} |e^{t\mathcal{A}} f(x)| &= \lim_{n\to\infty} |\mathcal{A}_{n,t} f(x)| \leq \limsup_{n\to\infty} \|\mathcal{A}_{n,t}\|_{\mathcal{L}} \, \|f\|_0 \\ &\leq M\, e^{\omega t} \|f\|_0, \;\; f \in \mathcal{C}_b(E), \; x \in E, \; t > 0, \end{aligned} \tag{6.4.29}$$

that allows us to state that $e^{t\mathcal{A}} \in \mathcal{L}(\mathcal{C}_b(E))$ and $\|e^{t\mathcal{A}}\|_{\mathcal{L}} \leq M\, e^{\omega t}$, $t \geq 0$.

In the sequel we prove that $e^{t\mathcal{A}}$ is a π-semigroup on $\mathcal{C}_b(E)$ with generator $\mathcal{A}$.

Step 4. We can rewrite assumption (v) in the following way. Let $(f_j) \subset \mathcal{C}_b(E)$ such that $f_j \xrightarrow{\pi} f$ as $j \to \infty$. For any $\epsilon > 0$, $x \in E$, $t_0 > 0$, there exists $j_0 = j_0(\epsilon, x, t_0)$ such that for any $j \geq j_0$, there exists a neighboorhood $\Lambda^j_{t_0}$ of t_0 for which it holds,

$$|e^{t\mathcal{A}} f_j(x) \;-\; e^{t\mathcal{A}} f(x)| \;\leq\; \epsilon, \quad t \in \Lambda^j_{t_0}. \tag{6.4.30}$$

From (6.4.30) we deduce in particular that $e^{t_0\mathcal{A}} f_j \xrightarrow{\pi} e^{t_0\mathcal{A}} f$ as $j \to \infty$ for any $t_0 \geq 0$.

Step 5. For any $f \in \mathcal{C}_b(E)$,

$$\lim_{t\to t_0} e^{t\mathcal{A}} f(x) \;=\; e^{t_0\mathcal{A}} f(x), \quad t_0 \geq 0, \;\; x \in E. \tag{6.4.31}$$

The case $t_0 = 0$ follows immediately by assumption (iii). Let us consider $t_0 > 0$. Fix $f \in \mathcal{C}_b(E)$, $x \in E$, $\epsilon > 0$. Choose a sequence $(f_j) \subset D(\mathcal{A}^2)$ such that $f_j \xrightarrow{\pi} f$

as $j \to \infty$. By step 4, there exists $j_0 = j_0(\epsilon, x, t_0) \geq 1$ and a neighboorhood Λ_{t_0} depending on j_0 so that it holds:

$$|e^{t\mathcal{A}}f(x) - e^{t\mathcal{A}}f_{j_0}(x)| \leq \epsilon, \quad t \in \Lambda_{t_0}.$$

Now (6.4.30) is proved, since taking into account also (3) of (6.4.18), there exists a neighboorhood $V_{t_0} \subset \Lambda_{t_0}$, depending on ϵ, x, t_0 such that it holds:

$$\begin{aligned} &|e^{t\mathcal{A}}f(x) - e^{t_0\mathcal{A}}f(x)| \leq |e^{t\mathcal{A}}f(x) - e^{t\mathcal{A}}f_{j_0}(x)| + \|e^{t\mathcal{A}}f_{j_0} - e^{t_0\mathcal{A}}f_{j_0}\|_0 \\ &+ |e^{t_0\mathcal{A}}f_{j_0}(x) - e^{t_0\mathcal{A}}f(x)| \leq 3\epsilon, \quad t \in V_{t_0}. \end{aligned} \tag{6.4.32}$$

Step 6. It holds: $e^{t\mathcal{A}}e^{s\mathcal{A}} = e^{(t+s)\mathcal{A}}$, $t, s \geq 0$.

Fix $f \in \mathcal{C}_b(E)$, $x \in E$, $s, t \geq 0$, $\epsilon > 0$. Choose a sequence $(f_j) \subset D(\mathcal{A}^5)$, $f_j \xrightarrow{\pi} f$ as $j \to \infty$. Let us remark that $e^{s\mathcal{A}}f_j \xrightarrow{\pi} e^{s\mathcal{A}}f$ as $j \to \infty$ and so using step 4 there exists $\hat{j} = \hat{j}(\epsilon, t, s, x)$ for which:

$$|e^{t\mathcal{A}}(e^{s\mathcal{A}}f_{\hat{j}})(x) - e^{t\mathcal{A}}(e^{s\mathcal{A}}f)(x)| + |e^{(t+s)\mathcal{A}}f_{\hat{j}}(x) - e^{(t+s)\mathcal{A}}f(x)| \leq \epsilon. \tag{6.4.33}$$

Since the semigroup law is satisfied on $D(\mathcal{A}^5)$ (see (6.4.19)) we find:

$$\begin{aligned} &|e^{t\mathcal{A}}e^{s\mathcal{A}}f(x) - e^{(t+s)\mathcal{A}}f(x)| \leq |e^{t\mathcal{A}}e^{s\mathcal{A}}f(x) - e^{t\mathcal{A}}e^{s\mathcal{A}}f_{\hat{j}}(x)| \\ &+ |e^{(t+s)\mathcal{A}}f_{\hat{j}}(x) - e^{(t+s)\mathcal{A}}f(x)| \leq \epsilon. \end{aligned}$$

For the arbitrariness of $\epsilon > 0$ and $x \in E$, we get $e^{t\mathcal{A}}e^{s\mathcal{A}}f = e^{(t+s)\mathcal{A}}f$ and so the semigroup law holds on $\mathcal{C}_b(E)$.

By the above arguments $e^{t\mathcal{A}}$ turns out to be a π-semigroup on $\mathcal{C}_b(E)$.

Step 7. Denote by $\mathcal{T}$ the generator of $e^{t\mathcal{A}}$, we prove that $\mathcal{T} = \mathcal{A}$.

By formula (6.4.22) we know that for any $f \in D(\mathcal{A}^3)$, $\frac{d}{dt}e^{t\mathcal{A}}f = e^{t\mathcal{A}}\mathcal{A}f$, $t \geq 0$. Proceeding as in step 10 of Cerrai [14, §5.1], for any $f \in D(\mathcal{A}^3)$, $\lambda > \omega$, $x \in E$ we integrate by parts,

$$\begin{aligned} &R(\lambda, \mathcal{T})\mathcal{A}f(x) = \int_0^\infty e^{-\lambda u}e^{u\mathcal{A}}\mathcal{A}f(x)du = \int_0^\infty e^{-\lambda u}\frac{d}{du}e^{u\mathcal{A}}f(x)\,du \\ &= -f(x) + \lambda\int_0^\infty e^{-\lambda u}e^{u\mathcal{A}}f(x)\,du = -f(x) + \lambda R(\lambda, \mathcal{T})f(x). \end{aligned} \tag{6.4.34}$$

Now consider any $g \in D(\mathcal{A})$. By (6.4.26) we know that $n^{2n}R(n^2)^n g \xrightarrow{\pi} g$ as $n \to \infty$. Moreover $n^{2n}R(n^2)^n g \in D(\mathcal{A}^3)$ for $n > \max(2, \omega)$ and so from (6.4.34) we obtain for any $\lambda > \omega$, $x \in E$,

$$\begin{aligned} &\int_0^\infty e^{-\lambda u}e^{u\mathcal{A}}\, n^{2n}R(n^2)^n\mathcal{A}g(x)\,du \\ &= \lambda\int_0^\infty e^{-\lambda u}e^{u\mathcal{A}}n^{2n}R(n^2)^n g(x)du - n^{2n}R(n^2)^n g(x). \end{aligned} \tag{6.4.35}$$

Letting $n \to \infty$ in (6.4.35) yields, by the Dominated Convergence Theorem,

$$R(\lambda, \mathcal{T})\,(\lambda g - \mathcal{A}g)(x) = g(x), \quad \lambda > \omega, \; x \in E.$$

This entails $g \in D(\mathcal{T})$ and $\mathcal{A}g = \mathcal{T}g$. Thus $\mathcal{T}$ extends $\mathcal{A}$. Moreover since $\rho(\mathcal{T}) \cap \rho(\mathcal{A}) \neq \emptyset$ we can conclude that $\mathcal{T} = \mathcal{A}$.

The proof is complete. ■

6.5 Final Remarks

We have presented the theory of π-semigroups in the space $\mathcal{C}_b(E)$ for convenience. However it is possible to extend this theory to more general spaces of functions. Here we briefly indicate how to proceed.

Let $B(E)$ be the Banach space of all bounded real functions on E, endowed with the sup norm. We can consider any linear subspace $\Theta(E)$ of $B(E)$ that satisfies the following two properties.

(i) $\Theta(E)$ is closed in $B(E)$ (with respect to the norm topology).
(ii) For any $T > 0$, for any map $G : [0,T] \times E \to \mathbb{R}$ satisfying:
- (a) $G(\cdot, x)$ is a Borel map on $[0,T]$ for any $x \in E$;
- (b) $G(s, \cdot) \in \Theta(E)$ for any $s \in [0,T]$;
- (c) $\sup_{s \in [0,T]} \|G(s,\cdot)\|_0 < \infty$,

we have that the map $g : E \to \mathbb{R}$, $g(x) = \int_0^T G(s,x)ds$, $x \in E$, belongs to $\Theta(E)$.

Conditions (i) and (ii) are similar to those introduced in Dynkin [31, page 57]. Moreover the space $\mathcal{C}_b(E)$ verifies these assumptions (see Lemma 6.2.8).

By (i), it follows that $(\Theta(E), \|\cdot\|_0)$ is a Banach space. On $\Theta(E)$ we can define π-convergence for sequences as in $\mathcal{C}_b(E)$ and also π-semigroups of bounded linear operators (through Definition 6.2.1 with $\mathcal{C}_b(E)$ replaced by $\Theta(E)$). Let P_t be a π-semigroup on $\Theta(E)$ of type ω. The following two basic facts about P_t can be deduced by (i) and (ii).

(1) For any $f \in \Theta(E)$, $T > 0$, the map $x \mapsto \int_0^T P_t f(x) dt$ belongs to $\Theta(E)$.
(2) For any $f \in \Theta(E)$, for any $\lambda > \omega$ the map g,

$$g(x) = \int_0^\infty e^{-\lambda u}\, P_u f(x)\, du, \quad x \in E, \quad \text{belongs to } \Theta(E).$$

To verify assertion (2), define the map $g_T(x) = \int_0^T e^{-\lambda u} P_u f(x) du$, $x \in E$ for any $T > 0$. First remark that $g_T \in \Theta(E)$, $T > 0$, and further $g \in B(E)$, since $\|P_u\|_{\mathcal{L}(\Theta(E))} \le M\ e^{\omega u}$, $u \ge 0$.
Then we have that $\|g - g_T\|_0$ tends to 0 as $T \to \infty$. Now, since $\Theta(E)$ is closed in $B(E)$, we conclude that $g \in \Theta(E)$.

Clearly in order to obtain (1) and (2) it is enough to assume, in hypothesis (a) of (ii), that the map $G(\cdot, x)$ is continuous on $[0,T]$ for any $x \in E$. Our generality is motivated by considering future applications to the Cauchy problem for $\pi-$semigroups, see the variation of constants formula in Section 7.2.

We emphasize that $\Theta(E)$ can be also the space $\mathcal{BC}(E)$ of all continuous, real and bounded functions on E. In particular all results of Section 6.2 and Section 6.3 can be easily adapted to the space $\mathcal{BC}(E)$. We point out that the heat and the Ornstein-Uhlenbeck semigroups, see Section 6.3, are π-semigroups on $\mathcal{BC}(H)$ as well.

The semigroup P_t, associated with the Dirichlet problem considered in §4.2, is not a π-semigroup on $\mathcal{BC}(\overline{H_+})$ (the same happens with the space $\mathcal{C}_b(\overline{H_+})$, compare with Remark 6.2.5). P_t is a π-semigroup on the Banach space of all functions in $\mathcal{BC}(H_+)$, which can be extended to maps belonging to $\mathcal{BC}(\overline{H_+})$, endowed with the sup norm.

We only check that U_t is a π-semigroup on $\mathcal{BC}(H)$. This is an immediate consequence of the following result.

Proposition 6.5.1 *For any $f \in \mathcal{BC}(H)$ one has*

$$\lim_{h\to 0} \ |U_{r+h}f(x) - U_r f(x)| \ = \ 0, \quad x \in H, \ r \geq 0. \tag{6.5.1}$$

Proof. Fix $f \in \mathcal{BC}(H)$, $x \in H$ and $r \geq 0$. We have

$$|U_t f(x) - U_r f(x)| \leq \int_H |f(S_t x + y) - f(S_r x + y)| \mathcal{N}(0, M(t))dy$$

$$+ \Big| \int_H f(S_r x + y)\mathcal{N}(0, M(t))dy - \int_H f(S_r x + y)\mathcal{N}(0, M(r))dy \Big|$$

$$= \ A^1(t) \ + \ A^2(t).$$

Now $\lim_{t\to r} A^2(t) = 0$. To see this fact notice that $\mathcal{N}(0, M(t))$ weakly converges to $\mathcal{N}(0, M(r))$ as $t \to r$, since $\lim_{t\to r} \|M(t) - M(r)\|_{\mathcal{L}_1(H)} = 0$, see Proposition 1.1.5. Of course if $r = 0$, $\mathcal{N}(0, M(t))$ weakly converges to the Dirac measure δ_0 as $t \to 0^+$.

As for $A^1(t)$, let us remark that the family of measures $\mathcal{N}(0, M(t))$, $t \in [0, r+1]$, is tight (see Cerrai [14, §6.3]). Hence for any $\eta > 0$, we can choose a compact set C in H such that $\mathcal{N}(0, M(t))\ (H \backslash C) < \eta$ for any $t \in [0, r+1]$. We obtain that

$$A^1(t) \leq \int_C |f(S_t x + y) - f(S_r x + y)| \mathcal{N}(0, M(t))dy$$

$$+ 2\eta \|f\|_0, \qquad t \in [0, r+1].$$

Now consider that the map : $[0, r+1] \times C \ \to \mathbb{R}$, $(t, y) \mapsto f(S_t x + y)$ is uniformly continuous. Hence there exists $\delta > 0$ such that $|t - r| < \delta$ implies $A^1(t) \leq \eta\,(2\|f\|_0 + 1)$. Thus we have $\lim_{t\to r} A^1(t) = 0$. The proof is complete. ■

Chapter 7

The Cauchy problem for a class of Markov type semigroups

We study the Cauchy problem for a class of Markov-type semigroups, see Chapter 6, in the space of all real, uniformly continuous and bounded functions on a separable metric space. In this class there are many transition Markov semigroups corresponding to stochastic differential equations in infinite dimensions as the heat semigroup and the one of Ornstein-Uhlenbeck. We define appropriate notions of solution and give existence and uniqueness theorems. Additional regularity results about the Cauchy problem associated with the Ornstein-Uhlenbeck semigroup are also proved.

7.1 Introduction

In Chapter 6 we have considered a new class of semigroups of bounded linear operators on $\mathcal{C}_b(E)$, ([1]) not strongly continuous in general. These semigroups are called $\pi-$semigroups, see Definition 6.2.1, and have been studied in Priola [67], [68], [69]. They are a development of weakly continuous semigroups introduced in Cerrai [14] (see also Cerrai and Gozzi [15]). This chapter is devoted to studying the Cauchy problem for π-semigroups (see Priola [67], Priola [69]).
We recall that the theory of π-semigroups is introduced in order to study transition semigroups of Markov Processes corresponding to the solutions of Stochastic Differential Equations and representing the solutions of PDE's with infinitely many variables, see (0.1.1) and (0.1.2).

We are concerned with the following Cauchy problem

$$\begin{cases} \partial_t u(t,x) \;=\; \mathcal{A}u(t,x) \;+\; F(t,x), \quad t \in]0,T],\ x \in E, \\ \\ u(0,x) = \;f(x), \quad\;\; x \in E, \end{cases} \tag{7.1.1}$$

where $\mathcal{A}$ is the generator of a π-semigroup (see Definition 6.2.6), $F : [0,T] \times E \to \mathbb{R}$ and $f \in \mathcal{C}_b(E)$. Let us notice that equations like (0.1.2) can be written in the more general form (7.1.1).

[1] $\mathcal{C}_b(E)$ denotes the Banach space of all real, uniformly continuous and bounded functions on a separable metric space E, endowed with the sup norm.

In Section 7.2, we define appropriate notions of solution for (7.1.1), by introducing classical, strict and strong solutions. Existence and uniqueness theorems for these solutions are also proved. In particular we are able to give conditions for uniqueness of the classical solution and show that this solution can be represented by the variation of constants formula (see Theorem 7.2.5). This leads to a new uniqueness result, see Theorem 7.2.7, for the mild solution of a Cauchy problem, involving a large class of Ornstein-Uhlenbeck operators in $\mathcal{C}_b(H)$ (see equation (0.1.2) where $Q(x) = Q$, $x \in H$, and also Cerrai and Gozzi [15]). The statements of our results are quite natural extensions of the classical ones, considered in the theory of $\mathcal{C}_0$ semigroups (see for instance §4 of Pazy [61]). However the proofs are involved and require new arguments.

In several applications, given a transition π-semigroup in $\mathcal{C}_b(\Omega)$ (Ω stands for an open subset of H) with generator $\mathcal{A}$, there exists a "natural" subspace of $D(\mathcal{A})$ where $\mathcal{A}$ can be represented as a "concrete" differential operator. We will denote this restriction of $\mathcal{A}$ by $\mathcal{A}_0$.

Given the initial value problem (7.1.1), it is useful to approximate any strong solution of (7.1.1) by means of a sequence u_n of strict solutions such that $u_n(t,\cdot) \in D(\mathcal{A}_0)$, $t \in [0,T]$, $n \geq 1$, see for instance Gozzi [38]. A similar problem has been investigated in Cerrai and Gozzi [15] for some classes of semigroups. In section 7.3 we present a different and more general approach to solve the problem, see Theorem 7.3.4 and Theorem 7.3.5. This can be applied in various situations, see Section 7.4.

The last section, see §7.4, is devoted to some important applications. We apply the previous results to Cauchy problems associated with the heat and the Ornstein-Uhlenbeck semigroups on $\mathcal{C}_b(H)$ and also with the semigroup corresponding to an infinite dimensional Dirichlet problem in a half space of H, see Chapter 5.

As concerns the Ornstein-Uhlenbeck semigroup U_t in $\mathcal{C}_b(H)$, a "natural" restriction of its generator is given by the differential operator $\mathcal{U}_0$, defined as follows, compare with equation (0.1.2),

$$\mathcal{U}_0 f(x) \;=\; \frac{1}{2}\mathrm{Tr}\,[MD^2 f(x)] + \;< A^* Df(x), x >, \quad x \in H, \tag{7.1.2}$$

where M is a self-adjoint, non negative and bounded linear operator on H, A generates a $\mathcal{C}_0$ semigroup on H (A^* denotes the adjoint of A) and f is suitably regular (see Definition 7.4.6).

As a consequence of Theorems 7.3.4 and 7.3.5, we are able to approximate the strong solution of a Cauchy problem, associated with U_t, by means of a sequence of strict solutions for which the operator $\mathcal{U}_0$ is well defined, see Theorem 7.4.7. This result extends Theorem 5.8 in Cerrai and Gozzi [15] and allows to give a meaning to the Ito formula that is used in applications like the study of second order Hamilton-Jacobi equations, arising from stochastic control theory (see for instance Gozzi [38], Gozzi and Rouy [39]).

7.2 Existence and uniqueness theorems

Let P_t be a π-semigroup of type ω on $\mathcal{C}_b(E)$ and let $\mathcal{A}$ be its generator. This section is devoted to studying the following initial value problem for a fixed $T > 0$:

$$\begin{cases} \partial_t u(t,x) \;=\; \mathcal{A}u(t,x) \;+\; F(t,x), \quad t \in]0,T],\ x \in E, \\ \\ u(0,x) = \; f(x), \qquad x \in E, \end{cases} \tag{7.2.1}$$

where E is a separable metric space, $f \in \mathcal{C}_b(E)$ and F satisfies suitable assumptions. We set $\mathcal{A}u(t,x) = \mathcal{A}u(t,\cdot)(x),\ x \in E, t \in]0,T]$. In the sequel we will use indifferently the symbols ∂_1 and ∂_t to denote the partial derivative with respect to the time variable. In order to treat (7.2.1) we introduce appropriate functions spaces.

Definition 7.2.1 Let I be an interval of $\mathbb{R}$ and $G : I \times E \to \mathbb{R}$ be a map, we say that:

(i) $G \in \mathcal{C}_\pi(I;\mathcal{C}_b(E))$ if $G(t,\cdot) \in \mathcal{C}_b(E)$ for any $t \in I$, $G(\cdot,x) : I \to \mathbb{R}$ is continuous for any $x \in E$ and

$$\|G\|_0 \;=\; \sup_{t\in I}\sup_{x\in E} |G(t,x)| \;<\; \infty.$$

(ii) $G \in \mathcal{C}^1_\pi(I;\mathcal{C}_b(E))$ if $G \in \mathcal{C}_\pi(I;\mathcal{C}_b(E))$, $G(\cdot,x) : I \to \mathbb{R}$ is continuously differentiable for any $x \in E$, $\partial_t G(t,\cdot) \in \mathcal{C}_b(E)$ for any $t \in I$ and

$$\|\partial_t G\|_0 \;=\; \sup_{t\in I}\sup_{x\in E} |\partial_t G(t,x)| \;<\; \infty.$$

(iii) $G \in \mathcal{C}_\pi(I;D(\mathcal{A}))$ if $G \in \mathcal{C}_\pi(I;\mathcal{C}_b(E))$, $G(t,\cdot) \in D(\mathcal{A})$ for any $t \in I$, $\mathcal{A}G(\cdot,x) : I \to \mathbb{R}$ is continuous for any $x \in E$ and

$$\|\mathcal{A}G\|_0 \;=\; \sup_{t\in I}\sup_{x\in E} |\mathcal{A}G(t,x)| \;<\; \infty.$$

Let us remark that, for any $T > 0$, if $G \in \mathcal{C}^1_\pi(]0,T[,\mathcal{C}_b(E)) \cap \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$ then it is easy to verify that G is continuous on $[0,T] \times E$.

We define the notion of π_I -convergence. Let $(G_n) \subset \mathcal{C}_\pi(I,\mathcal{C}_b(E))$, we say that G_n is π_I-*convergent* to $G \in \mathcal{C}_\pi(I,\mathcal{C}_b(E))$ and we write $G_n \xrightarrow{\pi_I} G$ as $n \to \infty$, if the following conditions hold:

$$\lim_{n\to\infty} G_n(t,x) = G(t,x) \quad t \in I,\ x \in E \quad \text{and} \quad \sup_{n\ge 1} \|G_n\|_0 < \infty. \tag{7.2.2}$$

We simply write π_T instead of $\pi_{[0,T]}$, for any $T > 0$. ∎

Having in mind the standard theory of parabolic problems for strongly continuous semigroups (see Pazy [61] and also Lunardi [55]), we make precise the notions of solution of (7.2.1).

Definition 7.2.2 Consider the problem (7.2.1) with the initial datum $f \in \mathcal{C}_b(E)$ and $F \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$. Then

(a) a map $u \in \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(E)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{A}))$ that satisfies (7.2.1) is said to be a *strict solution* of (7.2.1);

(b) a map $u \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$ is said to be a *strong solution* of (7.2.1), if there exists a sequence $(u_n) \subset \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(E)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{A}))$ such that

$$\begin{cases} u_n \xrightarrow{\pi_T} u, \quad \partial_t u_n - \mathcal{A}u_n \xrightarrow{\pi_T} F \ \text{ as } \ n \to \infty, \\ u_n(0,\cdot) \xrightarrow{\ \pi\ } f \quad \text{as } \ n \to \infty. \end{cases} \tag{7.2.3}$$

Let now $F \in \mathcal{C}_\pi(]0,T];\mathcal{C}_b(E))$. Then

(c) a map $u \in \mathcal{C}^1_\pi(]0,T];\mathcal{C}_b(E)) \cap \mathcal{C}_\pi(]0,T];D(\mathcal{A})) \cap \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$ that satisfies (7.2.1) is said to be a *classical solution* of (7.2.1). ∎

Clearly any strict solution of (7.2.1) is also a classical solution. We stress that for any $g \in D(\mathcal{A})$, the map $u(t,x) = P_t g(x)$ is a strict solution of (7.2.1) with the initial datum $f = g$ and $F = 0$. This follows readily by Proposition 6.2.7.

We are going to prove a uniqueness result about classical solutions. We need two preliminary lemmas that will be frequently used in the sequel.

Lemma 7.2.3 *Let I be an interval of $\mathbb{R}$, μ be a Borel finite measure on I and (X,d) be a separable metric space. Consider the functions $G,\ G_n : I \times X \to \mathbb{R}$, $n \geq 1$ that satisfy the following conditions:*

(i) $G_n(\cdot,x)$ is a Borel mapping for any $x \in X$, $n \geq 1$;

(ii) $G_n(t,\cdot)$ is a continuous mapping, for $n \geq 1$, $t \in I$;

(iii) there exists $g : I \to \mathbb{R}$, μ- integrable, such that $|G_n(t,x)| \leq g(t)$, $n \geq 1$, $x \in X$, $t \in I$;

(iv) $\lim_{n\to\infty} \sup_{x\in X} |G_n(t,x) - G(t,x)| = 0$, for $t \in I$.

Then we have

$$\lim_{n\to\infty} \sup_{x\in X} \int_I |G_n(t,x) - G(t,x)|\, \mu(dt) = 0.$$

Proof First let us notice that by assumptions (ii) and (iii), applying the Dominated Convergence Theorem, we find that the map $G(t,\cdot) : X \to \mathbb{R}$ is continuous and bounded for $t \in I$. Fix a countable dense subset D in X. We get

$$\sup_{x\in X} |G_n(t,x) - G(t,x)| = \sup_{x\in D} |G_n(t,x) - G(t,x)|, \quad n \geq 1,$$

since $|G_n(t,\cdot) - G(t,\cdot)|$ is continuous from X into $\mathbb{R}$, for $n \geq 1$, $t \in I$. Moreover for any $n \geq 1$, the map: $I \to \mathbb{R}$, $t \mapsto \sup_{x\in D} |G_n(t,x) - G(t,x)|$ is Borel and further $\sup_{x\in D} |G_n(t,x) - G(t,x)| \leq 2g(t)$, $n \geq 1$, $t \in I$. Thus we have

$$\sup_{x\in X} \int_I |G_n(t,x) - G(t,x)|\, \mu(dt) \leq \int_I \sup_{x\in D} |G_n(t,x) - G(t,x)|\, \mu(dt). \tag{7.2.4}$$

Letting $n \to \infty$ in the right-hand side of (7.2.4), by the Dominated Convergence Theorem, we get the assertion. ∎

Lemma 7.2.4 *Let $\mathcal{A}$ be the generator of a π-semigroup P_t on $\mathcal{C}_b(E)$ of type ω. For any $f \in \mathcal{C}_b(E)$ we have that $\lambda R(\lambda, \mathcal{A})f \xrightarrow{\pi} f$ as $\lambda \to \infty$.*

Moreover if P_t satisfies condition (6.2.4) of Definition 6.2.1 with respect to a non trivial covering $\mathcal{S}$ of E, then we have

$$\lim_{\lambda\to\infty} \sup_{x\in S} |\lambda R(\lambda, \mathcal{A})f(x) - f(x)| = 0, \quad f \in \mathcal{C}_b(E), \ S \in \mathcal{S}. \tag{7.2.5}$$

Proof Fix $f \in \mathcal{C}_b(E)$, invoking Proposition 6.2.11, we have

$$\|\lambda R(\lambda, \mathcal{A})f\|_0 \le \frac{K\lambda}{\lambda-\omega}\|f\|_0 \le 2K\|f\|_0, \quad \lambda > 2\omega. \tag{7.2.6}$$

Then fix $x \in E$ and for any $\lambda > \omega$,

$$\begin{aligned} &|\lambda R(\lambda, \mathcal{A})f(x) - f(x)| \le \lambda \int_0^\infty e^{-\lambda v}\,|P_v f(x) - f(x)|\,dv \\ &= \int_0^\infty e^{-w}\,|P_{\frac{w}{\lambda}} f(x) - f(x)|\,dw. \end{aligned} \tag{7.2.7}$$

Since for any $\lambda > 2\omega$, it holds

$$|P_{\frac{w}{\lambda}} f(x) - f(x)| \le 2M\|f\|_0\, e^{\frac{\omega\, w}{\lambda}} \le 2M\|f\|_0\, e^{\frac{w}{2}}, \quad w \in [0,\infty[\tag{7.2.8}$$

Letting $\lambda \to \infty$ in the last term of (7.2.7), we find $\lim_{\lambda\to\infty} \lambda R(\lambda, \mathcal{A})f(x) = f(x)$, by the Dominated Convergence Theorem. Thus the first assertion is proved.

Now suppose that P_t satisfies in addition condition (6.2.4). We write

$$\sup_{x\in S} |\lambda R(\lambda, \mathcal{A})f(x) - f(x)| \le \sup_{x\in S} \int_0^\infty e^{-w}\,|P_{\frac{w}{\lambda}} f(x) - f(x)| dw, \ \ S \in \mathcal{S}. \tag{7.2.9}$$

Notice that, for any $w \in [0,\infty[$, $\lim_{\lambda\to\infty} \sup_{x\in S} |P_{\frac{w}{\lambda}} f(x) - f(x)| = 0$.

Using the estimate (7.2.8) and invoking Lemma 7.2.3, we obtain that the right-hand side of (7.2.9) tends to 0 as $\lambda \to \infty$ and so also the last assertion is verified. ∎

Theorem 7.2.5 *Consider the initial value problem (7.2.1) and suppose that $f \in \mathcal{C}_b(E)$ and $F \in \mathcal{C}_\pi(]0,T]; \mathcal{C}_b(E))$. Then the problem has at most one classical solution. Further if it has a classical solution u, this solution is given by*

$$u(t,x) = P_t f(x) + \int_0^t P_{t-s}\,[F(s,\cdot)](x)ds = P_t f(x) + \int_0^t P_{t-s}F(s,x)ds, \quad t \in [0,T],\ x \in E. \tag{7.2.10}$$

Proof First let us notice that for any $x \in E,\ t > 0$, the map:

$$]0,t] \to \mathbb{R}, \quad s \mapsto P_{t-s}F(s,x) = P_{t-s}[F(s,\cdot)](x)$$

in general is not continuous on $]0,t]$. An example is given at the end of the theorem. However we are going to show that it is a Borel and bounded map and so the integral in (7.2.10) is meaningful in the Lebesgue sense.

Consider the map: $\phi :]0,t]\times]0,t] \to \mathbb{R}$, $\phi(p,q) = P_{t-p}F(q,x)$, $p,q \in]0,t]$. We claim that ϕ is separately continuous in each variable. The continuity with respect to p (with q fixed) is clear by the properties of P_t. As for the continuity with respect to q, remark that for any $q \in]0,t]$, $F(q+h,\cdot) \xrightarrow{\pi} F(q,\cdot)$ as $h \to 0$ and so $\lim_{h\to 0} P_{t-p}F(q+h,x) = P_{t-p}F(q,x)$. Thus ϕ is a Borel map on $]0,t]\times]0,t]$ and consequently $s \mapsto \phi(s,s)$ is a Borel map on $]0,t]$.

Moreover since $\|P_{t-s}F(s,\cdot)\|_0 \le Me^{\omega t}\|F\|_0$, $s \in]0,t]$, the map $s \mapsto P_{t-s}F(s,x)$ is also bounded. Applying Lemma 4.2.1, we get that the integral in (7.2.10) defines a function that belongs to $\mathcal{C}_b(E)$, for any $t \in [0,T]$.

Let now $u(t,x)$ be a classical solution of (7.2.1). Fix $t \in]0,T]$ and $z \in E$ and consider the map:

$$[0,t] \to \mathbb{R}, \quad s \mapsto P_{t-s}u(s,z) \tag{7.2.11}$$

In general this map is not differentiable (an example is provided at the end of the theorem), so we can not proceed as in the theory of strongly continuous semigroups (see for instance Corollary 4.2.2 of Pazy [61]).

However we will see that the resolvent operator $R(\lambda,\mathcal{A}) = R(\lambda)$, $\lambda > \omega$, applied to the mapping $P_{t-s}u(s,\cdot)$, has a regularizing effect not only on the spatial variable x but also on the time variable s. Indeed we are going to prove that for a fixed $\lambda > \omega$ the mapping:

$$\eta : [0,t] \to \mathbb{R}, \quad \eta(s) \stackrel{\text{def}}{=} R(\lambda)\, P_{t-s}u(s,z) = R(\lambda)\,[P_{t-s}u(s,\cdot)](z) \tag{7.2.12}$$

is continuous on $[0,t]$ and differentiable on $]0,t]$, having a bounded derivative. From this fact we will deduce (7.2.10). We have, by changing variable,

$$\begin{aligned}
&\eta(s) = R(\lambda)\, P_{t-s}u(s,z) = \int_0^\infty e^{-\lambda w}\, P_w(P_{t-s}u(s,z))\, dw \\
&= \int_0^\infty e^{-\lambda w}\, P_{w+t-s}u(s,z)\, dw = e^{\lambda t}e^{-\lambda s}\int_{t-s}^\infty e^{-\lambda v}\, P_v u(s,z)\, dv \\
&= g(s,s,s), \qquad \text{where} \quad g : [0,t]^3 \to \mathbb{R}, \\
&g(r_1,r_2,r_3) = e^{\lambda t}e^{-\lambda r_1}\int_{t-r_3}^\infty e^{-\lambda v}\, P_v u(r_2,z)\, dv, \quad r_i \in [0,t],\ i = 1,2,3.
\end{aligned} \tag{7.2.13}$$

Next computations are devoted to studying differentiability properties of g in order to obtain that η is differentiable in $]0,t]$ (the continuity of η in $s = 0$ will follow by similar arguments).

Claim 1. $\partial_1 g : [0,t]^3 \to \mathbb{R}$ is continuous.
We verify that $\partial_1 g$ is continuous in each variable, uniformly with respect to the others. We have, for any $r_1, r_2, r_3 \in [0,t]$ and h sufficiently small,

$$\begin{aligned} &\sup_{r_2\in[0,t],\, r_3\in[0,t]} |\partial_1 g(r_1+h, r_2, r_3) - \partial_1 g(r_1, r_2, r_3)| \\ &\le \lambda M \|u\|_0 \, e^{\lambda t} \, |e^{-\lambda(r_1+h)} - e^{-\lambda r_1}| \int_0^\infty e^{-v(\lambda-\omega)}\, dv, \end{aligned} \tag{7.2.14}$$

that tends to 0 as $h \to 0$. Further for any $r_1, r_2, r_3 \in [0,t]$,

$$\begin{aligned} &|\partial_1 g(r_1, r_2+h, r_3) - \partial_1 g(r_1, r_2, r_3)| \\ &\le \lambda e^{\lambda t} \int_0^\infty e^{-\lambda v} \, |P_v u(r_2+h, z) - P_v u(r_2, z)| \, dv. \end{aligned} \tag{7.2.15}$$

Now since $u \in \mathcal{C}_\pi([0,T]; \mathcal{C}_b(E))$, we have $u(r_2+h, \cdot) \xrightarrow{\pi} u(r_2, \cdot)$ as $h \to 0$ and so $\lim_{h\to 0} P_v u(r_2+h, z) = P_v u(r_2, z)$, $v \ge 0$; further $|P_v u(s,z)| \le M \|u\|_0 e^{\omega v}$, $v \ge 0$.

Letting $h \to 0$ in (7.2.15), we find by the Dominated Convergence Theorem,

$$\lim_{h\to 0} \sup_{r_1\in[0,t],\, r_3\in[0,t]} |\partial_1 g(r_1, r_2+h, r_3) - \partial_1 g(r_1, r_2, r_3)| = 0.$$

Finally we consider for any $r_1, r_2, r_3 \in [0,t]$ and h sufficiently small,

$$\begin{aligned} &|\partial_1 g(r_1, r_2, r_3+h) - \partial_1 g(r_1, r_2, r_3)| \\ &\le \lambda e^{\lambda t} \Big| \int_{t-r_3-h}^{t-r_3} e^{-\lambda v} \, P_v u(r_2, z) \, dv \Big| \le \lambda M \, e^{\lambda t} \, |h| \, \|u\|_0, \end{aligned} \tag{7.2.16}$$

that tends to 0 as $h \to 0$, uniformly in r_1, r_2, $r_3 \in [0,t]$. Now combining estimates (7.2.14), (7.2.15), (7.2.16), we obtain claim 1.

Claim 2. There exists $\partial_2 g$ on $[0,t] \times]0,t] \times [0,t]$ and it is a continuous function on $[0,t] \times]0,t] \times [0,t]$.

Since $u \in \mathcal{C}^1_\pi(]0,T]; \mathcal{C}_b(E))$, we have

$$\Big\| \frac{u(s+h, \cdot) - u(s, \cdot)}{h} \Big\|_0 \le \|\partial_1 u\|_0, \quad s \in [0,t],\ h \text{ small enough.}$$

Hence, using that P_t preserves π-convergence, we obtain that there exists $\partial_s P_v u(s,z) = P_v \partial_s u(s,z)$, $s \in]0,t]$, $v \ge 0$. Moreover since $|P_v \partial_s u(s,z)| \le M \, \|\partial_1 u\|_0 e^{\omega v}$, $s \in]0,t]$, $v \ge 0$, we can differentiate with respect to r_2 in the last integral of (7.2.13) and obtain

$$\partial_2 g(r_1, r_2, r_3) = e^{\lambda t} e^{-\lambda r_1} \int_{t-r_3}^\infty e^{-\lambda v} \, P_v \partial_{r_2} u(r_2, z) dv, \quad r_1, r_3 \in [0,t],\ r_2 \in]0,t].$$

Now arguing as for claim 1, we get claim 2.

Claim 3. There exists $\partial_3 g$ on $[0,t]^3$ and further $\partial_3 g : [0,t]^3 \to \mathbb{R}$ is continuous.

Indeed we have $\partial_3 g(r_1, r_2, r_3) = e^{-\lambda(r_1 - r_3)} P_{t-r_3} u(r_2, z)$, for $r_1,\ r_2,\ r_3 \in [0,t]$. Now the continuity of $\partial_3 g$ shall follow by proving that the map:

$$[0,t] \times [0,t] \to \mathbb{R}, \quad (r_2, r_3) \mapsto P_{t-r_3} u(r_2, z), \tag{7.2.17}$$

is continuous. To see this fact, first observe that the map is separately continuous in each variable on $[0,t]^2$. Then consider the estimate

$$|\partial_{r_3}(P_{t-r_3} u(r_2, z))| = |P_{t-r_3} \mathcal{A} u(r_2, z))| \leq M\, e^{\omega t} \|\mathcal{A} u\|_0,\ (r_2, r_3) \in [0,t]^2.$$

Thus also claim 3 is proved. We revert to the map $\eta(s) = g(s,s,s)$ defined in (7.2.12). By claim 1, claim 2 and claim 3, we derive that η is continuously differentiable on $]0,t]$ and that

$$\frac{d}{ds}\eta(s) = \partial_1 g(s,s,s) + \partial_2 g(s,s,s) + \partial_3 g(s,s,s), \quad s \in]0,t]. \tag{7.2.18}$$

Remark that one can prove the continuity of η in $s = 0$, proceeding in three steps as before. By (7.2.18) we get, for any $s \in]0,t]$,

$$\begin{aligned} \frac{d}{ds} R(\lambda) P_{t-s} u(s,z) &= -\lambda e^{\lambda t} e^{-\lambda s} \int_{t-s}^{\infty} e^{-\lambda v}\, P_v u(s,z)\, dv \\ &+ P_{t-s} u(s,z) + e^{\lambda t} e^{-\lambda s} \int_{t-s}^{\infty} e^{-\lambda v}\, P_v \partial_s u(s,z)\, dv \\ &= -\lambda R(\lambda) P_{t-s} u(s,z) + P_{t-s} u(s,z) + R(\lambda) P_{t-s} \partial_s u(s,z). \end{aligned} \tag{7.2.19}$$

From this formula, using that u is a solution of the initial value problem (7.2.1) and the identity: $\mathcal{A} R(\lambda, \mathcal{A}) = \lambda R(\lambda, \mathcal{A}) - I$, we obtain, for any $s \in]0,t]$,

$$\begin{aligned} \frac{d}{ds} R(\lambda) P_{t-s} u(s,z) &= -\lambda R(\lambda) P_{t-s} u(s,z) + P_{t-s} u(s,z) \\ &\quad + R(\lambda) \mathcal{A} P_{t-s} u(s,z) + R(\lambda) P_{t-s} F(s,z) \\ &= R(\lambda) P_{t-s} F(s,z). \end{aligned}$$

Then for any $\epsilon > 0$,

$$R(\lambda) u(t,z) - R(\lambda) P_{t-\epsilon} u(\epsilon, z) = \int_{\epsilon}^{t} R(\lambda) P_{t-s} F(s,z) ds.$$

Since the map $s \mapsto R(\lambda) P_{t-s} u(s,z)$ is continuous on $[0,t]$ and the map $s \mapsto R(\lambda) P_{t-s} F(s,z)$ is bounded on $]0,t]$, letting $\epsilon \to 0^+$ in the last formula, we obtain

$$R(\lambda) u(t,z) - R(\lambda) P_t f(z) = \int_0^t R(\lambda) P_{t-s} F(s,z) ds. \tag{7.2.20}$$

Multiplying both sides of (7.2.20) for λ we find

$$\lambda R(\lambda)\,[u(t,z)\;-\;P_t f(z)]\;=\;\int_0^t \lambda R(\lambda)P_{t-s}F(s,z)ds,\;\;\lambda>\omega. \tag{7.2.21}$$

We claim that letting $\lambda\to\infty$ in (7.2.21), we get formula (7.2.10). Indeed, invoking Lemma 7.2.4, the left-hand side of (7.2.21) tends to $u(t,z)\;-\;P_t f(z)$ as $\lambda\to\infty$. Let us consider the right-hand side. For any fixed $s\in]0,t]$, by Lemma 7.2.4, we have

$$\lim_{\lambda\to\infty}\lambda R(\lambda)P_{t-s}F(s,z)\;=\;P_{t-s}F(s,z).$$

Further $|\lambda R(\lambda)P_{t-s}F(s,z)|\le\lambda\|R(\lambda)\|_{\mathcal{L}}\|P_{t-s}F(s,\cdot)\|_0\le 2Me^{\omega t}\|F\|_0,\;\;s\in]0,t],\;\lambda>2\omega$. Applying the Dominated Convergence Theorem, the right-hand side of (7.2.21) tends to $\int_0^t P_{t-s}F(s,z)ds$ as $\lambda\to\infty$. This completes the proof. ■

The next examples clarify the proof of the previous theorem.

Examples 7.2.6 Let us consider the following π-semigroup P_t on $\mathcal{C}_b(\mathbb{R})$:

$$P_t f(x)=\;f(x-t),\;\;f\in\mathcal{C}_b(\mathbb{R}),\;x\in\mathbb{R}.$$

Let us fix $x\in\mathbb{R}$. Here we are concerned with the map $s\mapsto P_{1-s}F(s,x)=F(s,x+s-1)$, in case $F\in\mathcal{C}_\pi([0,1];\mathcal{C}_b(\mathbb{R}))$ or $F\in\mathcal{C}^1_\pi([0,1];\mathcal{C}_b(\mathbb{R}))\cap\mathcal{C}_\pi([0,1];D(\mathcal{A}))$. $\mathcal{A}$ denotes the generator of P_t, defined as follows:

$$D(\mathcal{A})\;=\;\mathcal{C}^1_b(\mathbb{R}),\quad \mathcal{A}f\;=\;-\frac{df}{dx},\;\;f\in D(\mathcal{A}).$$

Example (1) Set $a_k\;=\;3^{-k},\;k\ge 1$. We consider the following map $F:[0,1]\times\mathbb{R}\to\mathbb{R}$,

$$F(t,x)\;=\;\sum_{k=1}^{\infty}\frac{1}{2^k}\,\sin\Big(\frac{(t-a_k)\,(x-a_k)}{(t-a_k)^3\;+\;(x-a_k)^3}\Big),\quad t\ne a_k,\;x\ne a_k,\;k\ge 1,$$

$F(a_k,x)=\;F(t,a_k)\;=0,\;\;x\in\mathbb{R},\;t\in[0,1],\;k\ge 1$. We have that $F\in\mathcal{C}_\pi([0,1];\mathcal{C}_b(\mathbb{R}))$. Fix $x=1$ and consider the map $\gamma:[0,1]\to\mathbb{R}$,

$$\gamma(s)=\;P_{1-s}F(s,1)\;=F(s,s)=\sum_{k=1}^{\infty}\frac{1}{2^k}\,\sin\Big(\frac{1}{2}\,\frac{1}{s-a_k}\Big),\;\;s\ne a_k,$$

$\gamma(a_k)=0,\;\;k\ge 1$. It is straightforward to verify that γ is not continuous at every a_k, $k\ge 1$.

Example (2) Set $a_k\;=\;3^{-k},\;k\ge 1$. We consider the following map $u:[0,1]\times\mathbb{R}\to\mathbb{R}$,

$$u(t,x)\;=\;\sum_{k=1}^{\infty}\frac{1}{2^k}\,\sin\Big(\frac{(t-a_k)^2\,(x-a_k)^2}{[(t-a_k)^2\;+\;(x-a_k)^2]^{3/2}}\Big),\quad t\ne a_k,\;x\ne a_k,\;k\ge 1,$$

$u(a_k,x)\;=\;u(t,a_k)\;=\;0,\;\;x\in\mathbb{R},\;t\in[0,1],\;k\ge 1$. It is possible to verify that $u\in\mathcal{C}_\pi([0,1];\mathcal{C}_b(\mathbb{R}))\cap\mathcal{C}_\pi([0,1];D(\mathcal{A}))$.

Fix $x = 1$ and consider the map $\eta : [0, 1] \to \mathbb{R}$,

$$\eta(s) = u(s, s) = \sum_{k=1}^{\infty} \frac{1}{2^k} \sin\Big(\frac{1}{2^{3/2}} |s - a_k|\Big), \quad s \neq a_k,$$

$\eta(a_k) = 0$, $k \geq 1$. It is straightforward to verify that η is not differentiable at each a_k, $k \geq 1$.

∎

Now we provide a result of existence and uniqueness for strong solutions.

Theorem 7.2.7 *Consider the initial value problem (7.2.1) and suppose that* $f \in \mathcal{C}_b(E)$ *and* $F \in \mathcal{C}_\pi([O, T]; \mathcal{C}_b(E))$. *Then there exists a unique strong solution* u *for (7.2.1) and further for any fixed* $t \in [0, T]$, $x \in E$, *we have*

$$u(t, x) = P_t f(x) + \int_0^t P_{t-s} F(s, x) ds, \tag{7.2.22}$$

Proof <u>Existence.</u> First we verify that $u \in \mathcal{C}_\pi([0, T]; \mathcal{C}_b(E))$. It is enough to consider the term:

$$v(t, x) = \int_0^t P_{t-s} F(s, x) ds, \quad t \in [0, T], \ x \in E. \tag{7.2.23}$$

Clearly $\|v\|_0 \leq MT\, e^{\omega T} \|F\|_0$. We fix $x \in E$ and prove that $v(\cdot, x)$ is continuous on $[0, T]$. The continuity of $v(\cdot, x)$ in $t = 0$ is clear so consider any $\hat{t} \in]0, T]$.

We set $P_\eta = 0$, $\eta < 0$. For the increment h sufficiently small we have

$$\begin{aligned} v(\hat{t} + h, x) - v(\hat{t}, x) &= \int_0^{\hat{t}+h} P_{\hat{t}+h-s} F(s, x) ds - \int_0^{\hat{t}} P_{\hat{t}-s} F(s, x) ds \\ &= \int_0^T [P_{\hat{t}+h-s} F(s, x) - P_{\hat{t}-s} F(s, x)] ds. \end{aligned} \tag{7.2.24}$$

Now for any $s \in [0, T]$, except for $s = \hat{t}$, we have that

$$\lim_{h \to 0} P_{\hat{t}+h-s} F(s, x) = P_{\hat{t}-s} F(s, x),$$

thus letting $h \to 0$ in the last term of (7.2.24), by the the Dominated Convergence Theorem, we obtain the continuity of $v(\cdot, x)$ in $\hat{t}$.

Let $R(\lambda) = R(\lambda, \mathcal{A})$, $\lambda > \omega$, we consider the following approximations for any $n > \omega$,

$$u_n(t, x) = nR(n) P_t f(x) + \int_0^t nR(n) P_{t-s} F(s, x) ds, \quad x \in E, \ t \geq 0. \tag{7.2.25}$$

We check that for $n > \omega$, $u_n \in \mathcal{C}^1_\pi([O, T]; \mathcal{C}_b(E)) \cap \mathcal{C}_\pi([O, T]; D(\mathcal{A}))$. We use the following facts:

(a) $R(\lambda) P_t f = P_t R(\lambda) f$, $f \in \mathcal{C}_b(E)$, $\lambda > \omega$, $t \geq 0$. (it follows easily from the equality $\mathcal{A} P_t g = P_t \mathcal{A} g$, $g \in D(\mathcal{A})$, $t \geq 0$., see Proposition 6.2.7)

(b) the map: $[0,t] \to \mathbb{R}$, $s \mapsto \lambda R(\lambda)P_{t-s}F(s,x)$ is continuous for any $x \in E$, $t \geq 0$. (it follows by the proof of Theorem 7.2.5).

To obtain regularity properties for u_n, first remark that $P_t nR(n)f \in \mathcal{C}^1_\pi([O,T];\mathcal{C}_b(E))$ $\cap \mathcal{C}_\pi([O,T];D(\mathcal{A}))$, $n > \omega$. Then let us consider directly the more difficult term

$$v_n(t,x) \;=\; \int_0^t nR(n)P_{t-s}\,F(s,x)ds, \quad x \in E,\; t \in [0,T],\; n > \omega. \tag{7.2.26}$$

Let us fix $n > \omega$ and $x \in E$. We already know by (7.2.24) that $v_n(\cdot,x)$ is continuous on $[0,T]$. We establish the differentiability of $v_n(\cdot,x)$, setting for short $F_n(s,x) = nR(n)F(s,x)$, $s \in [0,T]$, $x \in E$.

Fix any $t \in]0,T[$, we start to prove the existence of the right derivative of $v_n(\cdot,x)$ in t. To this purpose we write for any $h > 0$, sufficiently small,

$$\frac{v_n(t+h,x) - v_n(t,x)}{h} \;=\; \frac{1}{h}\Big(\int_0^{t+h} P_{t+h-s}F_n(s,x)ds - \int_0^t P_{t+h-s}\,F_n(s,x)ds\Big)$$

$$+\frac{1}{h}\Big(\int_0^t P_{t+h-s}F_n(s,x)ds \;-\; \int_0^t P_{t-s}\,F_n(s,x)ds\Big) \;=\; \Gamma_1(h) + \Gamma_2(h) \quad \text{where}$$

$$\Gamma_1(h) \;=\; \frac{1}{h}\int_t^{t+h} P_{t+h-s}F_n(s,x)ds, \quad \Gamma_2(h) \;=\; \int_0^t \Big(\frac{P_{t+h-s} - P_{t-s}}{h}\Big)F_n(s,x)ds. \tag{7.2.27}$$

As concerns Γ_2, taking into account that

$$\frac{d}{dt}\Big(P_{t-s}F_n(s,x)\Big) \;=\; \mathcal{A}P_{t-s}F_n(s,x) = P_{t-s}\mathcal{A}F_n(s,x), \quad s \in [0,t[, \tag{7.2.28}$$

and $|\partial_t P_{t-s}F_n(s,x)| \leq M\,\|\mathcal{A}nR(n)\|_{\mathcal{L}}\|F\|_0\,e^{\omega T}$, $s \in [0,t[$, applying the Dominated Convergence Theorem we obtain the following formula,

$$\lim_{h\to 0^+} \Gamma_2(h) \;=\; \int_0^t P_{t-s}\mathcal{A}F_n(s,x)ds. \tag{7.2.29}$$

Let us turn to Γ_1. Changing variable, first $t+h-s = w$ and then $rh = w$, we can write for $h > 0$ sufficiently small

$$\Gamma_1(h) \;=\; \frac{1}{h}\int_0^h P_w F_n(t+h-w,x)dw \;=\; \int_0^1 P_{rh}F_n(t+h-rh,x)dr. \tag{7.2.30}$$

Now one has

$$\lim_{h\to 0^+} P_{rh}F_n(t+h-rh,x) \;=\; F_n(t,x), \quad r \in [0,1]. \tag{7.2.31}$$

Indeed let us consider the map $\phi : [0,t] \times [0,t] \to \mathbb{R}$, $\phi(u,v) = P_u F_n(v,x)$. ϕ is separately continuous in each variable and further there exists the partial derivative $\partial_u\phi$ on $[0,t] \times [0,t]$. Since $\partial_u\phi$ is bounded on $[0,t] \times [0,t]$, we can easily conclude that ϕ is continuous on $[0,t] \times [0,t]$ and so (7.2.31) is verified.

Applying the Dominated Convergence Theorem in (7.2.30) we conclude that $\lim_{h\to 0^+} \ \Gamma_1(h) = F_n(t,x)$.

By virtue of the above computations we have verified that there exists the right derivative of $v_n(\cdot,x)$ on $]0,T[$:

$$\frac{d^+}{dt}v_n(t,x) \ = \ \int_0^t P_{t-s}\mathcal{A}nR(n)F(s,x)ds \ + \ nR(n)F(t,x). \tag{7.2.32}$$

Let us remark that the right-hand side of (7.2.32), in the variable t, is a continuous function on $[0,T]$ (this fact can be verified arguing as in formula (7.2.24)). Consequently for a well known lemma of Real Analysis, see for instance §2.1.2 in Pazy [61], we deduce that $v_n(\cdot,x)$ is differentiable on $[0,T]$, with the derivative given by (7.2.32). To prove that $v_n \in \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(E))$, it remains to verify that $\partial_t v_n(t,\cdot) \in \mathcal{C}_b(E)$, $t\in[0,T]$, $n>\omega$. But this fact follows invoking Lemma 4.2.1.

Now we check that $v_n \in \mathcal{C}_\pi([0,T];D(\mathcal{A}))$. To this end we remark that for a fixed $t \in]0,T]$, $n>\omega$ and $x\in E$,

$$P_h(v_n(t,\cdot)) \ = \ \int_0^t P_hP_{t-s}F_n(s,x)\,ds. \tag{7.2.33}$$

Indeed in wiew of the continuity of the map $s \mapsto nP_{t-s}R(n)F(s,x)$ on $[0,t]$ (see condition (b)), the integral in (7.2.33) is a $\pi-$ limit of Riemann sums, see (6.2.15). Now from (7.2.33) we obtain easily that $v_n(t,\cdot)\in D(\mathcal{A})$ and it holds

$$\mathcal{A}v_n(t,x) \ = \ \lim_{h\to 0^+}\Big(\frac{P_h - I}{h}\Big)\,v_n(t,\cdot)(x) \ = \ \int_0^t \mathcal{A}P_{t-s}F_n(s,x)\,ds. \tag{7.2.34}$$

By this formula it follows that $\mathcal{A}v_n(\cdot,x)$ is continuous on $[0,T]$. Further $\mathcal{A}v_n(t,\cdot) \in \mathcal{C}_b(E)$, $t\in[0,T]$, $n>\omega$, thanks to Lemma 4.2.1. Hence $v_n \in \mathcal{C}_\pi([0,T];D(\mathcal{A}))$.

Finally we prove that

$$u_n(0,\cdot) \xrightarrow{\pi} f, \quad u_n \xrightarrow{\pi_T} u, \quad \partial_t u_n - \mathcal{A}u_n \xrightarrow{\pi_T} F \ \text{ as } n\to\infty.$$

First we have easily that

$$\|u_n\|_0 \ \le \ 2M\,e^{\omega T}(\|f\|_0 + T\|F\|_0), \quad n>2\omega.$$

Then applying Lemma 7.2.4 and the Dominated Convergence Theorem we obtain that for any $t\in[0,T]$, $x\in E$,

$$\begin{aligned}
&\lim_{n\to\infty} u_n(0,x) \ = \ \lim_{n\to\infty} nR(n)f(x) \ = \ f(x)\\
&\lim_{n\to\infty} u_n(t,x) = \lim_{n\to\infty} nR(n)P_tf(x) + \int_0^t nR(n)P_{t-s}\,F(s,x)ds,\\
&= \ P_tf(x) \ + \ \int_0^t P_{t-s}\,F(s,x)ds \ = \ u(t,x).
\end{aligned} \tag{7.2.35}$$

Using (7.2.34) we have that $\partial_t u_n(t,x) - \mathcal{A}u_n(t,x) \ = \ nR(n)F(t,x)$, $t\in[0,T]$, $x\in E$. Then it is clear that $nR(n)F \xrightarrow{\pi_T} F$ as $n\to\infty$. The existence of a strong solution is proved.

Uniqueness. Suppose that w is a strong solution of the Cauchy problem 7.2.1 and let (w_n) be a sequence of approximating strict solutions for w.

Setting $G_n \stackrel{\text{def}}{=} \partial_t w_n - \mathcal{A} w_n$ and $g_n \stackrel{\text{def}}{=} w_n(0,\cdot)$, for any $n \geq 1$, w_n is the strict solution of the following initial value problem

$$\begin{cases} \partial_t w_n(t,x) \ = \ \mathcal{A} w_n(t,x) \ + \ G_n(t,x), \quad t \in]0,T], \ x \in E, \\ w_n(0,x) = \ g_n(x), \qquad x \in E, \end{cases}$$

Applying Theorem 7.2.5 we obtain that

$$w_n(t,x) \ = \ P_t g_n(x) \ + \int_0^t P_{t-s}\, G_n(s,x) ds, \ \ x \in E, \ t \in [0,T]. \tag{7.2.36}$$

By our assumptions, $g_n \xrightarrow{\pi} f$ so that we have $P_q g_n \xrightarrow{\pi} P_q f$ as $n \to \infty$ for any $q \geq 0$. Moreover $G_n \xrightarrow{\pi_T} F$ and hence, by the Dominated Convergence Theorem, we get

$$\lim_{n \to \infty} \int_0^t P_{t-s}\, G_n(s,x) ds \ = \ \int_0^t P_{t-s}\, F(s,x) ds, \ \ x \in E, \ t \in [0,T].$$

Since $w_n \xrightarrow{\pi_T} u$, it follows that $u(t,x) = \lim_{n\to\infty} w_n(t,x) = \ P_t f(x) + \int_0^t P_{t-s}\, F(s,x) ds$, $t \in [0,T]$, $x \in E$. The proof is complete. ∎

The next result shows that if one imposes additional conditions on f and F in the Cauchy problem (7.2.1), then the strong solution (7.2.22) becomes a strict solution. We have the following two results.

Theorem 7.2.8 *Consider the initial value problem (7.2.1) and suppose that $f \in D(\mathcal{A})$ and $F \in \mathcal{C}^1_\pi([O,T];\mathcal{C}_b(E))$. Then the strong solution u of (7.2.1) is a strict solution.*

Proof Let us remark that $f \in D(\mathcal{A})$ is a necessary condition in order to obtain that there exists a strict solution for (7.2.1).

We write for any $t \in [0,T]$, $x \in E$,

$$\begin{aligned} u(t,x) \ &= \ P_t f(x) \ + \int_0^t P_{t-s}\, F(s,x) ds \\ &= \ P_t f(x) \ + \ v(t,x). \end{aligned} \tag{7.2.37}$$

It is easy to verify that $P_t f \in \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(E)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{A}))$. Therefore we deal with the map v.

We already know that $v \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$, see (7.2.24). We will deduce differentiability for v by considering the approximating mappings

$$v_n(t,x) \ = \int_0^t nR(n) P_{t-s}\, F(s,x) ds \ = \int_0^t P_q\, F_n(t-q,x) ds, \tag{7.2.38}$$

where $R(n)$ stands for $R(n,\mathcal{A})$, $n > \omega$, $F_n = nR(n)F$, $x \in E$, $t \in [0,T]$. From the Theorem 7.2.7, we already know that $v_n \in \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(E))$, $n > \omega$.

Here we need a different representation for $\partial_t v_n$, based on the existence of $\partial_t F$, compare with (7.2.32). To this purpose we will make computations similar to the proof of Theorem 7.2.7. Fixing $t \in]0,T[$, and $n > \omega$ and $x \in E$ we write for any $h > 0$, sufficiently small,

$$\frac{v_n(t+h,x) - v_n(t,x)}{h} = \frac{1}{h}\Big(\int_0^{t+h} P_q F_n(t+h-q,x)dq - \int_0^t P_q\, F_n(t+h-q,x)dq\Big)$$

$$+\frac{1}{h}\Big(\int_0^t P_q F_n(t+h-q,x)dq - \int_0^t P_q\, F_n(t-q,x)dq\Big) = \Gamma_1(h) + \Gamma_2(h)$$

$$\text{where} \quad \Gamma_1(h) = \frac{1}{h}\int_t^{t+h} P_q F_n(t+h-q,x)dq,$$

$$\Gamma_2(h) = \int_0^t P_q\Big(\frac{F_n(t+h-q,x) - F_n(t-q,x)}{h}\Big)\, dq. \tag{7.2.39}$$

As concerns Γ_2, take into account that under our assumptions, $h^{-1}\, F(s+h,\cdot) - F(s,\cdot) \xrightarrow{\pi} \partial_1 F(s,\cdot)$ as $h \to 0^+$, for any $s \in [0,T]$. Since

$$|\partial_t F_n(s,x)| = |nR(n)\partial_t F(s,x)| \le 2\|\partial_t F\|_0, \quad s \in [0,T], \; n > 2\omega,$$

applying the Dominated Convergence Theorem we obtain that

$$\lim_{h\to 0^+} \Gamma_2(h) = \int_0^t P_q nR(n)\partial_1 F(t-q,x)dq. \tag{7.2.40}$$

Let us turn to Γ_1. We can argue similarly to (7.2.30) and (7.2.31) in order to obtain that $\lim_{h\to 0^+} \Gamma_1(h) = P_t F_n(0,x)$. Using the above computations, it holds for any $r \in [0,T]$,

$$\partial_1 v_n(r,x) = \int_0^r P_{r-s} nR(n)\partial_1 F(s,x)\, ds + nR(n)P_r F(0,x). \tag{7.2.41}$$

Formula (7.2.41) allows us to state that $v(\cdot,x)$ is differentiable on $[0,T]$. Indeed we already know that $\lim_{n\to\infty} v_n(r,x) = v(r,x)$, $r \in [0,T]$ and further it is clear that

$$\lim_{n\to\infty} \partial_1 v_n(r,x) = \int_0^r P_{r-s}\partial_1 F(s,x)\, ds + P_r F(0,x) = w(r,x), \; r \in [0,T].$$

Now by (7.2.41) we have the following estimate, for $n > 2\omega$,

$$\|\partial_1 v_n\|_0 \le 2M\, e^{\omega T}\, (T\|\partial_1 F\|_0 + \|F\|_0). \tag{7.2.42}$$

Take into account (7.2.42) and the fact that $w(\cdot,x)$ is continuos on $[0,T]$. Applying a basic lemma of Real Analysis, we can conclude that $v(\cdot,x)$ is differentiable on $[0,T]$ and further

$$\partial_1 v(r,x) = \int_0^r P_{r-s}\partial_1 F(s,x)\, ds + P_r F(0,x), \quad r \in [0,T]. \tag{7.2.43}$$

From this formula we deduce that $v \in \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(E))$.

It remains to prove that $u \in \mathcal{C}_\pi([0,T];D(\mathcal{A}))$ and u satisfies the initial problem (7.2.1). To this purpose setting $u_n(t,x) = v_n(t,x) + nR(n)P_t f(t,x), \quad t \in [0,T]$, $x \in E$, we obtain

$$\mathcal{A}u_n(t,x) = \partial_1 u_n(t,x) - nR(n)F(t,x), \ \ t \in [0,T], \ x \in E, \ n > \omega. \tag{7.2.44}$$

Since by the previous computations, $\partial_1 u_n(t,\cdot) \xrightarrow{\pi} \partial_1 u(t,\cdot)$ as $n \to \infty$, we have that

$$\mathcal{A}u_n(t,\cdot) \xrightarrow{\pi} \partial_1 u(t,\cdot) - F(t,\cdot), \ \ t \in [0,T].$$

Since $\mathcal{A}$ is a $\pi-$ closed operator we find that $u(t,\cdot) \in D(\mathcal{A})$ and moreover $\mathcal{A}u(t,x) = \partial_1 u(t,x) - F(t,x), \ \ x \in E, \ t \in [0,T]$. The proof is complete. ∎

Proposition 7.2.9 *Consider the initial value problem (7.2.1) and suppose that* $f \in D(\mathcal{A})$ *and* $F \in \mathcal{C}_\pi([O,T];D(\mathcal{A}))$. *Then the strong solution* u, *see (7.2.22), is a strict solution.*

Proof Using the notation of Theorem 7.2.7 and Theorem 7.2.8, we write for any $t \in [0,T], \ \ x \in E$,

$$\begin{aligned} u(t,x) &= P_t f(x) + \int_0^t P_{t-s} F(s,x)ds \\ &= P_t f(x) + v(t,x). \end{aligned} \tag{7.2.45}$$

Since $P_t f \in \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(E)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{A}))$, it is enough to study regularity properties of the map v. Fix $x \in E$.

We already know that $v \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$, see (7.2.24). As in the proof of Theorem 7.2.8, we will deduce the differentiability of v by considering the approximating mappings

$$v_n(t,x) = \int_0^t nR(n)P_{t-s} F(s,x)ds, \quad n > \omega, \ t \in [0,T]. \tag{7.2.46}$$

We have by formula (7.2.32) and taking into account that $F(s,\cdot) \in D(\mathcal{A}), \ \ s \in [0,T]$,

$$\begin{aligned} \partial_1 v_n(t,x) &= \int_0^t P_{t-s}\mathcal{A}nR(n)F(s,x)ds + nR(n)F(t,x) \\ &= \int_0^t P_{t-s}nR(n)\mathcal{A}F(s,x)\,ds + nR(n)F(t,x), \ \ t \in [0,T], \ n > \omega. \end{aligned} \tag{7.2.47}$$

By the last formula we easily find the following estimate, for any $n > 2\omega$,

$$\|\partial_1 v_n\|_0 \le M\|nR(n)\|_{\mathcal{L}}(Te^{\omega T}\|\mathcal{A}F\|_0 + \|F\|_0) \le 2C_{T,\omega}(\|\mathcal{A}F\|_0 + \|F\|_0). \tag{7.2.48}$$

Since $\lim_{n\to\infty} v_n(r,x) = v(r,x)$, $r \in [0,T]$ and further

$$\lim_{n\to\infty} \partial_1 v_n(r,x) = \int_0^r P_{r-s}\mathcal{A}F(s,x)\,ds + F(r,x) = l(r,x), \ \ r \in [0,T],$$

Take into account (7.2.48) and the fact that $l(\cdot, x)$ is continuos on $[0, T]$. Applying a basic lemma of Real Analysis, see for instance §2.1.2 in Pazy [61], we can conclude that $v(\cdot, x)$ is differentiable on $[0, T]$ and further

$$\partial_1 v(r, x) \;=\; \int_0^r P_{r-s}\mathcal{A}F(s, x)\, ds + \; F(r, x), \quad r \in [0, T]. \tag{7.2.49}$$

From this formula we deduce that $v \in \mathcal{C}^1_\pi([0, T]; \mathcal{C}_b(E))$. Finally to check that $u \in \mathcal{C}_\pi([0, T]; D(\mathcal{A}))$ and satisfies the initial problem (7.2.1), we can proceed as in the end of the proof of Theorem 7.2.8, by using that $\mathcal{A}$ is a π-closed operator. This completes the proof. ■

7.3 A strong approximation result

In this section we only consider transition π-semigroups on $\mathcal{C}_b(\Omega)$, where Ω is any open subset of a real separable Hilbert space H (with inner product $< \cdot, \cdot >$ and norm $|\cdot|$).

We recall that $\mathcal{L}_1(H)$ denotes the Banach space of all trace class operators on H, endowed with the norm $\|T\|_1 \;=\; \mathrm{Tr}\,(\sqrt{T^*T}),\; T \in \mathcal{L}_1(H)$ (see Chapter 1).

Definition 7.3.1 We recall that $\mathcal{C}^k_b(\Omega),\; k \geq 1$, denotes the subspace of $\mathcal{C}_b(\Omega)$ of all functions having uniformly continuous and bounded Fréchet derivatives up to the order k. We introduce the following space

$$\tilde{\mathcal{C}}^2_b(\Omega) \stackrel{\text{def}}{=} \{ f \in \mathcal{C}^2_b(\Omega),\;\text{ such that }\; D^2 f \in \mathcal{C}_b(\Omega, \mathcal{L}_1(H))\, \}(^2). \tag{7.3.1}$$

$\tilde{\mathcal{C}}^2_b(\Omega)$ is a Banach space, endowed with the following norm:

$$\|f\|_{\tilde{2}} \;=\; \|f\|_2 + \sup_{x\in\Omega} \|D^2 f(x)\|_{\mathcal{L}_1(H)}, \;\; f \in \tilde{\mathcal{C}}^2_b(\Omega).$$

Throughout this section, $(\mathcal{D}_b(\Omega), \|\cdot\|_{\mathcal{D}})$ will denote *one* of the following two Banach spaces: $\mathcal{C}^1_b(\Omega)$ or $\tilde{\mathcal{C}}^2_b(\Omega)$. ▪

In several applications, given a transition π-semigroup on $\mathcal{C}_b(\Omega)$, with generator $\mathcal{A}$, there exists a "natural" subspace of $D(\mathcal{A})$ where $\mathcal{A}$ can be represented as a "concrete" differential operator that we denote by $\mathcal{A}_0$.

Let us consider the initial value problem (7.2.1) and suppose that a restriction $\mathcal{A}_0$ of $\mathcal{A}$ is given. It is useful for various applications (see for instance Gozzi [38] and Gozzi and Rouy [39]) to approximate any strong solution of (7.2.1), by means of a sequence u_n of strict solutions such that $u_n(t, \cdot) \in D(\mathcal{A}_0)$, $t \geq 0$, $n \geq 1$. This problem was investigated in Cerrai and Gozzi [15] for some classes of semigroups. Our approach is different and more general.

We make the following preliminary assumptions.

[2] Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces and let $S \subset E$. We denote by $\mathcal{C}_b(S, F)$ the Banach space of all uniformly continuous and bounded functions from S into F, endowed with the usual sup norm: $\|f\|_0 \;=\; \sup_{x\in S} \|f(x)\|_F$.

Hypothesis 7.3.2 Let P_t be a transition π-semigroup of type ω on $\mathcal{C}_b(\Omega)$, with generator $\mathcal{A}$. Let $\mathcal{A}_0 : D(\mathcal{A}_0) \subset \mathcal{C}_b(\Omega) \to \mathcal{C}_b(\Omega)$ be a linear operator. Suppose that $\mathcal{A}_0$ is a restriction of $\mathcal{A}$ and that it holds:

(i) $P_t \in \mathcal{L}(\mathcal{D}_b(\Omega)),\ t \geq 0$ and there exists a real Borel map $g \in L^1_{loc}([0,\infty[)$ such that $\|P_t\|_{\mathcal{L}(\mathcal{D}_b(\Omega))} \leq g(t),\ t \geq 0$;

$$(ii) \quad D(\mathcal{A}_0) \supset \bigcup_{\lambda>\omega} R(\lambda, \mathcal{A})\,(\mathcal{D}_b(\Omega)). \ \blacksquare \tag{7.3.2}$$

We need the following definition, that is introduced in Cerrai and Gozzi [15].

Definition 7.3.3 A sequence $(f_n) \subset \mathcal{C}_b(\Omega)$ is said to be $\mathcal{K}$- *convergent* to a map $f \in \mathcal{C}_b(\Omega)$ and we shall write $f_n \xrightarrow{\mathcal{K}} f$ as $n \to \infty$ if

$$\begin{array}{l} \sup\limits_{n\geq 1} \|f_n\|_0 < \infty \ \text{ and for any compact subset } K \subset \Omega, \\ \lim\limits_{n\to\infty} \sup\limits_{x\in K} |f_n(x) - f(x)| = 0. \end{array} \tag{7.3.3}$$

Similarly let $(G_n) \subset \mathcal{C}_\pi([0,T], \mathcal{C}_b(\Omega))$, we say that G_n is $\mathcal{K}_T$- *convergent* to a map $G \in \mathcal{C}_\pi([0,T], \mathcal{C}_b(\Omega))$ and we shall write $G_n \xrightarrow{\mathcal{K}_T} G$ as $n \to \infty$ if $\sup_{n\geq 1} \|G_n\|_0 < \infty$ and moreover for any compact set $K \subset \Omega$ one has:

$$\lim_{n\to\infty} \sup_{x\in K,\, t\in[0,T]} |G_n(t,x) - G(t,x)| = 0. \tag{7.3.4}$$

Finally let $S \subset \mathcal{C}_b(\Omega)$. S is said to be $\mathcal{K}$- *dense* (respectively $\pi-$ dense) in $\mathcal{C}_b(\Omega)$ if for any $f \in \mathcal{C}_b(\Omega)$, there exists a sequence $(f_n) \subset S$ such that f_n $\mathcal{K}$- converges to f (respectively $f_n \xrightarrow{\pi} f$). ∎

Now we state the two main results of the section.

Theorem 7.3.4 *Consider the initial value problem (7.2.1), associated with a transition π-semigroup P_t on $\mathcal{C}_b(\Omega)$, with generator $\mathcal{A}$. Suppose that $f \in \mathcal{C}_b(\Omega)$ and $F \in \mathcal{C}_\pi([O,T]; \mathcal{C}_b(\Omega))$. Let $\mathcal{A}_0$ be a restriction of $\mathcal{A}$. Assume that P_t and $\mathcal{A}_0$ verify the assumptions of Hypothesis 7.3.2 with respect to $\mathcal{D}_b(\Omega)$.*

Then, denoting by u the strong solution of (7.2.1), there exists a sequence (u_n) of strict solutions such that:

$$\begin{array}{l} (i)\ u_n \xrightarrow{\pi_T} u, \quad \partial_t u_n - \mathcal{A} u_n \xrightarrow{\pi_T} F \text{ as } n \to \infty; \\ (ii)\ u_n(t,\cdot) \in D(\mathcal{A}_0), \quad t \geq 0, \ \ n \geq 1. \end{array} \tag{7.3.5}$$

The next result shows that if the datum F of (7.2.1) is more regular and P_t satisfies an additional assumption then the strong solution can be approximated in a better way.

More precisely, we consider a π-semigroup P_t, satisfying the following additional condition, for any compact subset K in Ω,

$$\lim_{h\to 0} \sup_{x\in K} |P_{t+h} f(x) - P_t f(x)| = 0, \ \ f \in \mathcal{C}_b(\Omega), \ t \geq 0. \tag{7.3.6}$$

Theorem 7.3.5 *Consider the initial value problem (7.2.1) associated with a transition π-semigroup P_t on $\mathcal{C}_b(\Omega)$, satisfying in addition condition (7.3.6). Suppose that $f \in \mathcal{C}_b(\Omega)$, $F \in \mathcal{C}_\pi([O,T];\mathcal{C}_b(\Omega))$ and further that F is continuous on $[0,T] \times \Omega$.*

Let $\mathcal{A}_0$ be a restriction of the generator $\mathcal{A}$ of P_t and assume that P_t and $\mathcal{A}_0$ verify the assumptions of Hypothesis 7.3.2 with respect to $\mathcal{D}_b(\Omega)$.

Then, denoting by u the strong solution of (7.2.1), there exists a sequence (u_n) of strict solutions such that:

$$\begin{array}{l} (i)\ \ u_n \xrightarrow{\mathcal{K}_T} u, \quad \partial_t u_n - \mathcal{A} u_n \xrightarrow{\mathcal{K}_T} F \text{ as } \ n \to \infty; \\ (ii)\ \ u_n(t,\cdot) \in D(\mathcal{A}_0), \quad t \geq 0, \ \ n \geq 1. \end{array} \tag{7.3.7}$$

The remainder of this section is devoted to the proof of the previous two results.

Moreover in the next section we provide a simpler proof of Theorems 7.3.4 and 7.3.5, in case P_t is the heat semigroup in $\mathcal{C}_b(H)$. This way we can illustrate better the ideas of the proofs without the technical difficulties of the general case. The proofs require some tools that we introduce now.

First we construct, following Da Prato [20], a very regularizing Ornstein-Uhlenbeck semigroup on $\mathcal{C}_b(H)$.

Fix an orthonormal basis $\{e_k\}_{k\geq 1}$ of H. For any $x \in H$, set $x_k =< x, e_k >$, $k \geq 1$. Fix a sequence (μ_k) such that $\mu_k < 0$, $k \geq 1$ and $-\sum_{k=1}^{\infty} \frac{1}{\mu_k} < \infty$.

Define the following linear operator $\tilde{B} : D(\tilde{B}) \subset H \to H$, where

$$D(\tilde{B}) = \{x \in H, \text{ such that } \sum_{k=1}^{\infty} \mu_k^2 x_k^2 < \infty\}, \ \ \tilde{B}x \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \mu_k x_k \, e_k. \tag{7.3.8}$$

$\tilde{B}$ is a one to one, self-adjoint, negative operator. Moreover there exists $\tilde{B}^{-1} \in \mathcal{L}_1(H)$, hence in particular, $\tilde{B}$ is a closed operator. It is straightforward to verify that $D(\tilde{B})$ is dense in H. For the Lumer- Phillips Theorem, $\tilde{B}$ generates a $\mathcal{C}_0$-semigroup of contractions on H, which we will be denote by $e^{t\tilde{B}}$, $t \geq 0$. It can be also show, by using arguments of Spectral Theory, see Da Prato [20], that $e^{t\tilde{B}}$ is an analytic semigroup on H and moreover that each operator $e^{t\tilde{B}}$ is Hilbert-Schmidt on H, see Chapter 1, for any $t > 0$.

Now we define a linear operator $Q_t : H \to H$, $t \geq 0$.

$$Q_t x \ = \int_0^t e^{s\tilde{B}} e^{s\tilde{B}^*} x \, ds \ = \ \frac{1}{2}\Big(e^{2t\tilde{B}} \tilde{B}^{-1} x \ - \ \tilde{B}^{-1} x\Big), \ \ x \in H, \ t \geq 0.$$

It is easy to verify that Q_t is a self-adjoint, one to one, positive and trace class operator on H, for any $t > 0$. Finally we consider the regularizing Ornstein- Uhlenbeck semigroup associated with $\tilde{B}$, defined as follows:

$$\tilde{Z}_t f(x) \ = \int_H f(e^{t\tilde{B}} x \ + y) \mathcal{N}(0, Q_t) \, dy, \quad f \in \mathcal{C}_b(H), \ x \in H, \ t > 0. \tag{7.3.9}$$

Note that (7.3.9) is a special case of (6.3.9). Although $\tilde{Z}_t$ is not a strongly continuous semigroup on $\mathcal{C}_b(H)$ (see Cerrai [14] and §6.3), it enjoys some interesting properties.

We denote again by $\tilde{Z}_t$ the Ornstein-Uhlenbeck semigroup on $\mathcal{B}_b(H)$ ([3]), associated with $\tilde{B}$ (it is defined by formula (7.3.9) with $\mathcal{C}_b(H)$ replaced by $\mathcal{B}_b(H)$). It is known that $\tilde{Z}_t(\mathcal{B}_b(H)) \subset \mathcal{C}_b^n(H)$, for any $n \geq 1$ (the proof of this strong Feller property of $\tilde{Z}_t$ is given in Da Prato and Zabczyk [23, §9.19]). Moreover it holds

$$
\begin{aligned}
&(a) \text{ for any compact set } K \subset H, \ \lim_{h\to 0} \sup_{x\in K} |\tilde{Z}_{t+h}f(x) - \tilde{Z}_t f(x)| = 0, \\
&\quad f \in \mathcal{C}_b(H), \ t \geq 0. \\
&(b) \ \tilde{Z}_t(\mathcal{C}_b(H)) \subset \tilde{\mathcal{C}}_b^2(H) \text{ and } \tilde{Z}_t : \mathcal{C}_b(H) \to \tilde{\mathcal{C}}_b^2(H) \text{ is continuous }, \ t > 0 \\
&(\text{i.e. } \ \|\tilde{Z}_t f\|_{\tilde{2}} \leq C_t \, \|f\|_0, \ f \in \mathcal{C}_b(H), \ t \geq 0).
\end{aligned}
\tag{7.3.10}
$$

Statement (a) was first proved in Cerrai [14, §6.2, §6.3] (see also Proposition 6.3.3). Statement (b) is proved in Da Prato and Zabczyk [23, §9.20] and in Da Prato [20, §2.5].

By means of the semigroup $\tilde{Z}_t$ on $\mathcal{C}_b(H)$, we construct a family of regularizing linear operators on $\mathcal{C}_b(\Omega)$, for an arbitrary open subset Ω of H.

Definition 7.3.6 We consider an operator $N : \mathcal{C}_b(\Omega) \to \mathcal{B}_b(H)$, defined as follows:
If $\Omega = H$, N is the identity of $\mathcal{C}_b(\Omega)$;
If $\Omega \neq H$, for any $f \in \mathcal{C}_b(\Omega)$,

$$
Nf(x) = f(x) \text{ for } x \in \Omega, \quad Nf(x) = 0, \text{ for } x \notin \Omega. \tag{7.3.11}
$$

Clearly N is a linear and continuous operator. We introduce a family of operators $\{\tilde{U}_t\}_{t\geq 0} \subset \mathcal{L}(\mathcal{C}_b(\Omega))$, defined as follows

$$
\tilde{U}_t f(x) \overset{\text{def}}{=} \tilde{Z}_t(Nf)(x) = \int_H Nf(e^{t\tilde{B}}x + y)\mathcal{N}(0, Q_t)\, dy, \quad f \in \mathcal{C}_b(\Omega), \, x \in \Omega, \, t \geq 0. \tag{7.3.12}
$$

Notice that in general $\tilde{U}_t$ is not a semigroup of operators in $\mathcal{C}_b(\Omega)$ and $\tilde{U}_t = \tilde{Z}_t$ in $\mathcal{C}_b(H)$. We call $\tilde{U}_t$ the *Ornstein- Uhlenbeck approximations* (or briefly O-U approximations) on $\mathcal{C}_b(\Omega)$. ∎

The next lemma justifies our definition.

Lemma 7.3.7 *Let $\tilde{U}_t$ be the O-U approximations on $\mathcal{C}_b(\Omega)$, the following statements hold:*
(i) $\tilde{U}_t(\mathcal{C}_b(\Omega)) \subset \tilde{\mathcal{C}}_b^2(\Omega)$, and $\tilde{U}_t \in \mathcal{L}(\mathcal{C}_b(\Omega), \tilde{\mathcal{C}}_b^2(\Omega))$, $t > 0$;
(ii) there exists a family of Borel positive measures $\{q(t, x, \cdot)\}_{x\in\Omega, t\geq 0}$ such that: $q(t, x, \Omega) \leq 1$, $q(0, x, \cdot) = \delta_x$, $x \in \Omega$, $t \geq 0$ and

$$
\tilde{U}_t f(x) = \int_\Omega f(y)\, q(t, x, dy), \quad f \in \mathcal{C}_b(\Omega), \ x \in \Omega, \ t \geq 0; \tag{7.3.13}
$$

(iii) for any compact set $K \subset \Omega$, $\lim_{t\to 0^+} \sup_{x\in K} |\tilde{U}_t f(x) - f(x)| = 0$, $f \in \mathcal{C}_b(\Omega)$.

[3] $\mathcal{B}_b(H)$ denotes the Banach space of all real, bounded and Borel functions on H, endowed with the sup norm.

Proof (i) Using (b) of formula (7.3.10), we get for any $f \in \mathcal{C}_b(\Omega)$,

$$\tilde{U}_t f(x) \;=\; \tilde{Z}_t(Nf)(x) \;=\; \tilde{Z}_{t/2}[\tilde{Z}_{t/2}(Nf)](x), \;\; x \in \Omega.$$

Hence $\tilde{U}_t f \in \tilde{\mathcal{C}}_b^2(\Omega)$, $t > 0$. Moreover the continuity of $\tilde{U}_t$ follows since we have $\|\tilde{U}_t f\|_{\tilde{2}} \leq c_{t/2} \sup_{x\in H} |\tilde{Z}_{t/2} Nf(x)| \leq c_{t/2} \|f\|_0$.

(ii) We argue as in Proposition 6.2.16. Consider the family of linear positive functionals $\{q_{t,x}\}_{t\geq 0,\, x\in\Omega}$ on $\mathcal{C}_b(\Omega)$,

$$f \mapsto q_{t,x}(f) \stackrel{\text{def}}{=} \tilde{U}_t f(x), \quad f \in \mathcal{C}_b(\Omega),\; t \geq 0,\; x \in \Omega.$$

For any $(f_n) \subset \mathcal{C}_b(\Omega)$, such that $f_n \uparrow f$ (i.e. $f_n(x) \uparrow f(x)$, $x \in \Omega$), with $f \in \mathcal{C}_b(\Omega)$, it follows that $\tilde{U}_t f_n \uparrow \tilde{U}_t f$ for any $t \geq 0$, by the Monotone Convergence Theorem. Hence applying the Daniell Theory there exists for any $t \geq 0$, $x \in \Omega$, a Borel measure $q(t, x, \cdot)$ such that (7.3.13) holds.

(iii) Fix $f \in \mathcal{C}_b(\Omega)$, K compact subset in Ω and $\epsilon > 0$. First, there exists $\delta > 0$, such that for any x, $z \in \Omega$ with $|x - z| < \delta$ we have that $|f(x) - f(z)| < \epsilon$.

Then since K is a compact set, there exists $\eta > 0$ such that

$$\eta < \delta \;\text{ and }\; K^\eta = \cup_{x\in K} B(x,\eta) \,\subset\, \Omega, \tag{7.3.14}$$

where $B(x, \eta) = \{y \in H \text{ such that } |x - y| \leq \eta\}$. Since the semigroup $e^{t\tilde{B}}$ (used in (7.3.12)) is strongly continuous on H, it is straightforward to verify that there exists $\hat{t}$ for which $\sup_{x\in K} |e^{t\tilde{B}}x - x| \leq \eta/2$, $t < \hat{t}$. Hence for any $y \in H$ with $|y| \leq \eta/2$ and for any $x \in K$, we have that $|e^{t\tilde{B}}x - x + y| \leq \eta$, $t < \hat{t}$. Consequently it holds

$$e^{t\tilde{B}}x + y \,\in B(x, \eta) \subset\, K^\eta \subset \Omega, \quad \text{for any } \; x \in K,\; y \in B(0, \eta/2),\; t < \hat{t}. \tag{7.3.15}$$

Now taking into account (7.3.15) and proceeding similarly to Cerrai [14, §6.2] we can obtain for any $t < \hat{t}$:

$$\begin{aligned}
&\sup_{x\in K} |\tilde{U}_t f(x) - \, f(x)| \,\leq \sup_{x\in K} \int_H |\, Nf(e^{t\tilde{B}}x + y) - f(x)| \mathcal{N}(0, Q_t)dy \\
&\leq \sup_{x\in K} \int_{|y|<\frac{\eta}{2}} |f(e^{t\tilde{B}}x + y) - \, f(x)| \mathcal{N}(0, Q_t)\, dy \\
&+ 2\|f\|_0 \int_{|y|\geq\frac{\eta}{2}} \mathcal{N}(0, Q_t)\, dy \,\leq\, \epsilon + \frac{4}{\eta}\|f\|_0 \int_H |y|\, \mathcal{N}(0, Q_t)dy \\
&\leq\, \epsilon \,+\, \frac{4}{\eta}\|f\|_0 \sqrt{\mathrm{Tr}\; Q_t}.
\end{aligned} \tag{7.3.16}$$

Letting $t \to 0^+$ in the last term of (7.3.16), one deduces the assertion (iii), in virtue of the arbitrariness of ϵ. ∎

We will use the Prokhorov Theorem in the following form, see for instance Kuo [50].

Let M be a set of positive finite Borel measures on a separable and complete metric spaces X, the next conditions are equivalent:
(i) M is equibounded (i.e. $\sup_{\mu\in M}\mu(X)<\infty$) and tight (i.e. for any $\epsilon>0$ there exists a compact set $K_\epsilon\subset X$ such that $\sup_{\mu\in M}\mu\,(X\backslash K_\epsilon)<\epsilon$).
(ii) M is weakly relatively compact (i.e. for any sequence $(\mu_n)\subset M$ there exists a positive finite Borel measure ν and a subsequence (μ_k) such that μ_k converges weakly to ν, that is $\int_X f(y)\mu_k(dy)\to\int_X f(y)\nu(dy)$ as $k\to\infty$, for any $f\in\mathcal{C}_b(X)$).

Let us emphasize that the Prokhorov Theorem can be applied to Borel measures on Ω. Indeed there exists a metric d on Ω that induces the given topology of Ω and such that (Ω,d) is a complete separable metric space (see for instance Ash [4, §A9.8]).

For the proof of the main theorems, we prepare two lemmas. We recall that here $(\mathcal{D}_b(\Omega),\|\cdot\|_{\mathcal{D}})$ denotes $\mathcal{C}_b^1(\Omega)$ or $\tilde{\mathcal{C}}_b^2(\Omega)$.

Lemma 7.3.8 *Let $G:[0,T]\times\Omega\to\mathbb{R}$ be any map such that:*

- (i) $G(\cdot,x)$ is a Borel map for any $x\in\Omega$;
- (ii) $G(s,\cdot)\in\mathcal{D}_b(\Omega)$, $s\in[0,T]$;
- (iii) $\|G(s,\cdot)\|_{\mathcal{D}_b(\Omega)}\le g(s)$, $s\in[0,T]$, where g is a real Borel map on $[0,T]$ that is Lebesgue integrable.

(7.3.17)

Then for any fixed $t\in[0,T]$ the map $\phi:\Omega\to\mathbb{R}$,

$$\phi(x)=\int_0^t G(s,x)\,ds,\quad x\in\Omega\qquad\text{belongs to }\ \mathcal{D}_b(\Omega).$$

Proof Fix any $t\in]0,T]$. First let us observe that by Lemma 4.2.1, the mapping ϕ belongs to $\mathcal{C}_b(\Omega)$. Then the proof is divided into two parts.

Part 1. $\mathcal{D}_b(\Omega)=\mathcal{C}_b^1(\Omega)$.

Fix $x\in\Omega$, $v\in H$ and consider for any $r>0$:

$$\frac{\phi(x+rv)-\phi(x)}{r}=\int_0^t\frac{G(s,x+rv)-G(s,x)}{r}\,ds. \tag{7.3.18}$$

By assumption (ii), $r^{-1}\,|G(s,x+rv)-G(s,x)|\le\|D_xG(s,\cdot)\|_0|v|\le g(s)\,|v|$ for any $x\in\Omega$, $v\in H$, $s\in[0,t]$. Letting $r\to0^+$ in (7.3.18), by the Dominated Convergence Theorem, we obtain that ϕ is Gâteaux differentiable at $x\in\Omega$, with the Gâteaux derivative

$$<D\phi(x),v>=\int_0^t<D_xG(s,x),v>\,ds,\quad v\in H. \tag{7.3.19}$$

(notice that the map $s\mapsto<D_xG(s,x),v>$ is Borel for any $x\in\Omega$, $v\in H$). Now we check that $D\phi\in\mathcal{C}_b(\Omega,H)$, so that in particular $D\phi$ turns out to be also a Fréchet derivative and the proof of step 1 is complete.

The boundedness of $D\phi$ is evident, let us prove the uniform continuity. For any $v \in H$ such that $\|v\|_H = 1$, we claim that the following estimate holds:

$$
\begin{aligned}
&| < D\phi(x) - D\phi(z), v > | \;\le\; \int_0^t | < D_xG(s,x) - D_xG(s,z), v > | \, ds \\
&\le \int_0^t \|D_xG(s,x) - D_xG(s,z)\|_H \, ds \;\le\; 2\int_0^t g(s)\, ds, \quad x,\, z \in \Omega.
\end{aligned} \tag{7.3.20}
$$

Indeed, since H is separable, fix a countable dense subset D of the unit ball of H, then for any $s \in [0,t]$, $x,\, z \in \Omega$, using the continuity of $< D_xG(s,x), \cdot >$, we have: $\sup_{\|v\|_H=1} | < D_xG(s,x) - D_xG(s,z), v > | = \sup_{v\in D} | < D_xG(s,x) - D_xG(s,z), v > |$. Hence the map $s \mapsto \| < D_xG(s,x) - D_xG(s,z)\|_H$ is Borel. Moreover it holds:

$$
\|D_xG(s,x) - D_xG(s,z)\|_H \;\le\; 2\|D_xG(s,\cdot)\|_0 \le\; 2\|G(s,\cdot)\|_{\mathcal{D}} \;\le 2g(s),\;\; s \in [0,T] \tag{7.3.21}
$$

and so formula (7.3.20) is meaningful. From (7.3.20), we deduce the next useful formula:

$$
\|D\phi(x) - D\phi(z)\|_H \;\le\; \int_0^t \|D_xG(s,x) - D_xG(s,z)\|_H \, ds, \quad\; x,\, z \in \Omega. \tag{7.3.22}
$$

Now consider the following sets: $(\Omega - z) \;=\; \{y - z\}_{y\in\Omega}$ for any $z \in H$. The uniform continuity of $D\phi$ follows if we show that the map: $H \to \mathbb{R}$, $z \to \sup_{x\in(\Omega-z)\cap\Omega} \|D\phi(x+z) - D\phi(x)\|_H$ is continuous in $z = 0$. To this purpose it is enough to verify that for any sequence $(z_n) \subset H$ such that $z_n \to 0$, it holds:

$$
\lim_{n\to\infty} \sup_{x\in(\Omega-z_n)\cap\Omega} \|D\phi(x+z_n) \;-\; D\phi(x)\|_H \;=\; 0, \tag{7.3.23}
$$

Fix such a sequence (z_n) and a countable dense subset L_n of $(\Omega - z_n)\cap\Omega$, $n \ge 1$. For any fixed $s \in [0,t]$, $n \ge 1$, we have:

$$
\sup_{x\in(\Omega-z_n)\cap\Omega} \|D_xG(s,x+z_n) - D_xG(s,x)\|_H = \sup_{x\in L_n} \|D_xG(s,x+z_n) - D_xG(s,x)\|_H.
$$

Notice that the maps $\gamma_n : s \mapsto \sup_{x\in L_n} \|D_xG(s,x+z_n) - D_xG(s,x)\|_H$ are Borel, for any $n \ge 1$. Further we have $\gamma_n(s) \le 2g(s)$, $s \in [0,T]$, $n \ge 1$. Hence we can write

$$
\sup_{x\in(\Omega-z_n)\cap\Omega} \|D\phi(x+z_n) \,-\, D\phi(x)\|_H \le \int_0^t \sup_{x\in L_n} \|D_xG(s,x+z_n) - D_xG(s,x)\|_H \, ds.
$$

Letting $n \to \infty$, by the Dominated Convergence Theorem, we obtain (7.3.23) and the uniform continuity of $D\phi$ is proved.

Part 2. $\mathcal{D}_b(\Omega) \;=\; \tilde{\mathcal{C}}_b^2(\Omega)$.

From part 1, we already know that $\phi \in \mathcal{C}_b^1(\Omega)$ and (7.3.19) holds. First we prove that $D\phi : \Omega \to H$, is Gâteaux differentiable at any $x \in \Omega$. To this purpose, denote again by D a countable dense subset of the unit ball of H. We obtain for any $w \in H$, $x \in \Omega$,

$$
\begin{aligned}
&\sup_{\|v\|_H=1} \Big| < \frac{D\phi(x+rw) - D\phi(x)}{r}, v > - \int_0^t < D_x^2 G(s,x)(w), v > \, ds \Big| \\
&= \sup_{\|v\|_H=1} \Big| \int_0^t \frac{< D_x G(s,x+rw) - D_x G(s,x), v >}{r} ds - \int_0^t < D_x^2 G(s,x)(w), v > ds \Big| \\
&\le \sup_{\|v\|_H=1} \int_0^t \Big| < \frac{D_x G(s,x+rw) - D_x G(s,x)}{r} - D_x^2 G(s,x)(w), \, v > \Big| ds \\
&\le \int_0^t \sup_{v \in D} \Big| < \frac{D_x G(s,x+rw) - D_x G(s,x)}{r} - D_x^2 G(s,x)(w), \, v > \Big| ds, \quad r > 0.
\end{aligned}
\tag{7.3.24}
$$

Arguing as for (7.3.20), the last integral is meaningful. Letting $r \to 0^+$ in the last term of (7.3.24), by the Dominated Convergence Theorem, we obtain that there exists the Gâteaux derivative $D^2\phi(x)$ of $D\phi$ at $x \in H$ and it holds

$$
< D^2\phi(x)(w), v >= \int_0^t < D_x^2 G(s,x)(w), v > \, ds, \quad v, \, w \in H, \, x \in \Omega. \tag{7.3.25}
$$

At this point in order to derive the Fréchet differentiability of $D\phi$, it is enough to verify that the second Gâteaux derivative $D^2\phi \in \mathcal{C}_b(\Omega, \mathcal{L}(H))$. The boundedness of $D^2\phi$ is clear, we establish the uniform continuity. We have:

$$
\begin{aligned}
&\|D^2\phi(x) - D^2\phi(z)\|_{\mathcal{L}(H)} = \sup_{(u,v) \in D \times D} | < [D^2\phi(x) - D^2\phi(z)](u), v > | \\
&\le \int_0^t \sup_{(u,v) \in D \times D} | < \left[D_x^2 G(s,x) - D_x^2 G(s,z)\right](u), v > | \, ds \\
&= \int_0^t \|D_x^2 G(s,x) - D_x^2 G(s,z)\|_{\mathcal{L}(H)} \, ds \le 2 \int_0^t g(s) \, ds, \quad x, \, z \in \Omega.
\end{aligned}
\tag{7.3.26}
$$

From (7.3.26), proceeding as in part 1 for formula (7.3.23), it follows the uniform continuity of $D^2\phi$. Thus we have obtained that $\phi \in \mathcal{C}_b^2(\Omega)$.
It remains to prove that the second Fréchet derivative $D^2\phi \in \mathcal{C}_b(\Omega, \mathcal{L}_1(H))$.

Denote by $\|\cdot\|_1$ the norm of $\mathcal{L}_1(H)$. Let $\mathcal{T}$ the subspace of $\mathcal{L}(H)$ of all finite rank operators. We set $\mathcal{T}_1 = \{N \in \mathcal{T}$ such that $\|N\|_{\mathcal{L}(H)} \le 1\}$. We recall the following result, Lemma 1.1.3,

An operator $T \in \mathcal{L}(H)$ belongs to $\mathcal{L}_1(H)$ if and only if it holds: $\sup_{N \in \mathcal{T}_1} |\mathrm{Tr}\,(NT)| = M < \infty$. Moreover if $T \in \mathcal{L}_1(H)$, then $M = \|T\|_1$.
It is simple to check directly that $\mathcal{T}_1(H)$ is separable in $\mathcal{L}(H)$. Thus we can choose a countable dense subset $\mathcal{M}$ of $\mathcal{T}_1$. Now remark that for any $A \in \mathcal{L}_1(H)$ the linear map: $\mathcal{L}(H) \to \mathbb{R}$, $N \mapsto \mathrm{Tr}\,(NA)$ is continuous. Using these facts we conclude that:

$$
\|T\|_1 = \sup_{N \in \mathcal{T}_1} |\mathrm{Tr}\,(NT)| = \sup_{N \in \mathcal{M}} |\mathrm{Tr}\,(NT)|, \quad T \in \mathcal{L}_1(H). \tag{7.3.27}
$$

Appealing to (7.3.27), we prove that $D^2\phi(x) \in \mathcal{L}_1(H)$. We have for any $x \in \Omega$,

$$\begin{aligned}
\sup_{N\in\mathcal{T}_1} |\mathrm{Tr}\,(ND^2\phi(x))| &= \sup_{N\in\mathcal{T}_1} \Big|\int_0^t \mathrm{Tr}\,(ND_x^2G(s,x))\,ds\Big| \\
&\le \int_0^t \sup_{N\in\mathcal{M}} |\mathrm{Tr}\,(ND_x^2G(s,x))|\,ds \\
&= \int_0^t \|D_x^2G(s,x)\|_1 ds \;\le\; 2\int_0^t g(s)\,ds, \quad x\in\Omega.
\end{aligned} \tag{7.3.28}$$

(the last integral is meaningful since $\mathcal{M}$ is countable and so the map $s \mapsto \|D_x^2G(s,x)\|_1$ is Borel). To prove the uniform continuity of $D^2\phi$, we use the following estimates that can be obtained arguing as in (7.3.28), for any $x,\, z \in \Omega$,

$$\begin{aligned}
&\|D^2\phi(x) - D^2\phi(z)\|_1 \;=\; \sup_{N\in\mathcal{T}_1} \Big|\int_0^t \mathrm{Tr}\,(N[D_x^2G(s,x) - D_x^2G(s,z)]\,)ds\Big| \\
&\le \int_0^t \|D_x^2G(s,z) \;-\; D_x^2G(s,z)\|_1\,ds.
\end{aligned} \tag{7.3.29}$$

From (7.3.29), arguing as in (7.3.23), it follows the desired uniform continuity of $D^2\phi$. The proof is complete. ■

We need the following technical result.

Lemma 7.3.9 *Let P_t be a transition π-semigroup on $\mathcal{C}_b(\Omega)$ of type ω, with generator $\mathcal{A}$. Then*

$$Y \;=\; \bigcup_{\lambda>\omega} \lambda R(\lambda,\mathcal{A})\,(\tilde{\mathcal{C}}_b^2(\Omega))$$

is π- dense in $\mathcal{C}_b(\Omega)$. Moreover if, for any compact set $K \subset \Omega$, P_t satisfies the following condition:

$$\lim_{t\to 0^+}\sup_{x\in K} |P_tf(x) - f(x)| \;=\; 0, \quad f\in\mathcal{C}_b(\Omega), \tag{7.3.30}$$

then Y is $\mathcal{K}$- dense in $\mathcal{C}_b(\Omega)$, see Definition 7.3.3.

Proof Fix $f \in \mathcal{C}_b(\Omega)$ and define $f_n \;=\; \tilde{U}_{1/n}f$, $n \ge 1$, where $\tilde{U}_t$, $t \ge 0$ are the O-U approximations on $\mathcal{C}_b(\Omega)$, introduced in Definition 7.3.6.
By Lemma 7.3.7, we known that $f_n \in \tilde{\mathcal{C}}_b^2(\Omega)$ and moreover f_n $\mathcal{K}-$ converges to f. We set $R(\lambda) = R(\lambda,\mathcal{A})$. Let us remark that for any $n > \omega$, we have that $nR(n)f_n \in Y$. Thus the first assertion follows if we prove that

$$nR(n)f_n \xrightarrow{\pi} f, \;\text{ as }\; n\to\infty. \tag{7.3.31}$$

Since $\|nR(n)f_n\|_0 \le 2\|\tilde{U}_{1/n}f\|_0 \le 2\|f\|_0$, $n > 2\omega$, to verify (7.3.31) it is enough to show that

$$\lim_{n\to\infty} |nR(n)f_n(x) \;-\; f(x)| \;=\; 0. \tag{7.3.32}$$

Fix $n > \omega$ and consider, for any fixed $x \in \Omega$,

$$\begin{array}{l} |nR(n)f_n(x) \,-\, f(x)| \leq \\ |nR(n)f_n(x) \,-\, nR(n)f(x)| \,+\, |nR(n)f(x) \,-\, f(x)| = \\ \Gamma^1(n,x) \,+\, \Gamma^2(n,x). \end{array} \tag{7.3.33}$$

By Lemma 7.2.4, we know that $\lim_{n\to\infty} \Gamma^2(n,x) = 0$. Let us consider the remainder term.

$$\begin{aligned} \Gamma^1(n,x) &\leq n\int_0^\infty e^{-nu}\,|P_u f_n(x) - P_u f(x)|\,du \\ &\leq \int_0^\infty e^{-v}\,|P_{\frac{v}{n}} f_n(x) - P_{\frac{v}{n}} f(x)|\,dv. \end{aligned} \tag{7.3.34}$$

Now we prove that for any $v > 0$, it holds:

$$\lim_{n\to\infty} |P_{v/n}\,(f_n - f)(x)| \,=\, 0. \tag{7.3.35}$$

Since $\|P_{v/n}\,(f_n - f)\|_0 \leq 2M\,e^{\omega\frac{v}{n}}\,\|f\|_0 \leq Ce^{\frac{v}{2}}$, $n > 2\omega$, once (7.3.35) is verified, letting $n \to \infty$ in the last term of (7.3.34), by the Dominated Convergence Theorem, we find $\lim_{n\to\infty} \Gamma^1(n,x) = 0$ and so formula (7.3.32) holds.

To prove (7.3.35), denote by $p(t,x,B)$ the transition Markov function associated with P_t, with $x \in \Omega$, $t \geq 0$ and B Borel set of Ω (see Definition 6.2.15). Since

$$\lim_{n\to\infty} P_{v/n} f(x) = f(x), \qquad v > 0,$$

we derive that, for any fixed $v > 0$, the sequence of measures $p(\frac{v}{n}, x, \cdot)$ converges weakly to δ_x as $n \to \infty$ (where δ_x stands for the Dirac measure concentrated at $x \in \Omega$). Applying the Prokhorov Theorem, for any $\epsilon > 0$, there exists a compact set $C_\epsilon \subset \Omega$ such that $p(\frac{v}{n}, x, \Omega\backslash C_\epsilon) < \epsilon$ for any $n \geq 1$. Thus we have for any $v > 0$,

$$\begin{array}{l} |P_{\frac{v}{n}}\,(f_n - f)(x)| \,\leq\, \displaystyle\int_\Omega |f_n(y) \,-\, f(y)|\,p(v\backslash n\,, x, dy) \\ \leq \displaystyle\int_{C_\epsilon} |f_n(y) \,-\, f(y)|\,p(v\backslash n, x, dy) \,+\, 2\int_{\Omega\backslash C_\epsilon} \|f - f_n\|_0\, p(v\backslash n, x, dy) \\ \leq \displaystyle\int_{C_\epsilon} |f_n(y) \,-\, f(y)|\,p(v\backslash n, x, dy) \,+\, 2\|f\|_0\, p(v\backslash n, x, \Omega\backslash C_\epsilon) \end{array} \tag{7.3.36}$$

For f_n $\mathcal{K}$- converges to f, we can choose n_0 such that $\sup_{x\in C_\epsilon} |f_n(x) - f(x)| < \epsilon$, for any $n \geq n_0$. Thus for any $n \geq n_0$,

$$|P_{\frac{v}{n}}\,(f_n - f)(x)| \,\leq\, \epsilon\,[1 \,+ 2\|f\|_0]$$

and (7.3.35) is established. Thus the first part of the lemma is proved.

Now suppose that P_t satisfies also condition (7.3.30). Then fix a compact set K of Ω. We need to prove that

$$\lim_{n\to\infty} \sup_{x\in K} |nR(n)f_n(x) \,-\, f(x)| \,=\, 0. \tag{7.3.37}$$

We proceed as for (7.3.32) with the same notations. First we have by Lemma 7.2.4 (with the covering $\mathcal{S}$ replaced by the family of all compact sets in Ω) $\lim_{n\to\infty}\sup_{x\in K}\,|nR(n)f(x) - f(x)| = 0$.
Let us prove that $\lim_{n\to\infty}\,\sup_{x\in K}\,\Gamma^1(n,x)$. We have

$$\sup_{x\in K}\Gamma^1(n,x) \le \sup_{x\in K}\int_0^\infty e^{-v}|P_{\frac{v}{n}}f_n(x) - P_{\frac{v}{n}}f(x)|\,dv.$$

We will apply the Dominated Convergence Theorem in the right-hand side of last formula (see Lemma 7.2.3 for more details) in order to obtain the assertion. To this purpose it is enough to check that for any $v > 0$, it holds:

$$\lim_{n\to\infty}\sup_{x\in K}|P_{v/n}\,(f_n - f)(x)| \,=\, 0. \tag{7.3.38}$$

This formula follows by estimates (7.3.36), if we show that the family of measures $\mathcal{P} = \{p(\frac{v}{n}, x, \cdot)\}_{n\ge 1,\, x\in K}$ is tight for $v > 0$ fixed. By the Prokhorov Theorem it suffices to prove that $\mathcal{P}$ is weakly relatively compact. To this purpose fix a sequence $(p(v/k, x_k, \cdot)) \subset \mathcal{P}$. Since K is a compact set there exists a subsequence (x_{k_j}) that converges to $x \in K$. Setting $k_j = j$, for convenience, we claim that $(p(v/j, x_j, \cdot))$ converges weakly to δ_x as $n \to \infty$. Indeed for any $f \in \mathcal{C}_b(\Omega)$,

$$\begin{aligned}\Big|\int_\Omega f(y)\,p(v\backslash j, x_j, dy) \;-\; f(x)\Big| &\le |P_{\frac{v}{j}}f(x_j) \;-\; f(x_j)| \;+\; |f(x_j) \;-f(x)|\\ \le\; \sup_{x\in K}|P_{\frac{v}{j}}f(x) \;-\; f(x)| \;&+\; |f(x_j) \;-f(x)|.\end{aligned} \tag{7.3.39}$$

On making $j \to \infty$ in the last term of (7.3.39), we find formula (7.3.38), by assumption (7.3.30). The proof is complete. ■

Proof of Theorem 7.3.4. **(i)** We set $R(\lambda) = R(\lambda, \mathcal{A})$ for any $\lambda > \omega$ and consider the following functions, for $x \in \Omega$, $n > \omega$, $t \ge 0$,

$$u_n(t,x) = P_t nR(n)\tilde{U}_{\frac{1}{n}}f(x) + \int_0^t P_{t-s}nR(n)\tilde{U}_{\frac{1}{n}}F(s,x)\,ds, \tag{7.3.40}$$

where $\tilde{U}_t$ stands for the O-U approximations on $\mathcal{C}_b(\Omega)$, introduced in Definition 7.3.6 and we write $\tilde{U}_{\frac{1}{n}}F(s,x) = \tilde{U}_{\frac{1}{n}}\;(F(s,\cdot))(x)$, $x \in \Omega$, $s \in [0,T]$.

$$\text{Set}\quad f_n \;=\; \tilde{U}_{\frac{1}{n}}f \quad\text{and}\quad F_n \;=\; \tilde{U}_{\frac{1}{n}}F,\;\; n \ge 1.$$

It is worth noticing that $F_n(\cdot, x)$ is continuous on $[0,T]$. To see this fact let us observe that for any $n \ge 1$, in light of Lemma 7.3.7, assertion (ii),

$$\tilde{U}_{\frac{1}{n}}F(t,x) \;=\; \int_\Omega F(t,y)\,q(1/n,\,x,dy),\;\;\; t \in [0,T],\; x \in \Omega,$$

where $q(1/n,\,x,\cdot)$ are finite Borel measures on Ω. Now the continuity in t is clear by the Dominated Convergence Theorem. Therefore $F_n \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(\Omega))$ and we can use the same arguments of the proof of Theorem 7.2.7 in order to check that $u_n \in \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(\Omega)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{A}))$.

Moreover we claim that F_n is continuous on $[0,T]\times\Omega$, $n\geq 1$. Indeed the maps $F_n(s,\cdot)\in \mathcal{C}^1_b(\Omega)$ (by Lemma 7.3.7) for $s\in[0,T]$, $n\geq 1$ and the following estimate holds

$$\sup_{s\in[0,T]} \|D_x F_n(s,\cdot)\|_0 \leq \sup_{s\in[0,T]} c_n \|F(s,\cdot)\|_0 = c_n\|F\|_0, \quad n\geq 1.$$

To prove that $u_n \xrightarrow{\pi_T} u$, as $n\to\infty$, let us consider formula (7.3.32) of Lemma 7.3.9, namely

$$nR(n)\tilde{U}_{\frac{1}{n}} f = nR(n)\, f_n \xrightarrow{\pi} f, \quad \text{as } n\to\infty, \quad f\in \mathcal{C}_b(\Omega). \tag{7.3.41}$$

By this formula, we derive that for any $t\geq 0$, $f\in\mathcal{C}_b(\Omega)$,

$$P_t nR(n)\, f_n \xrightarrow{\pi} P_t f, \quad \text{as } n\to\infty. \tag{7.3.42}$$

Then using the estimate: $|nR(n)\tilde{U}_{\frac{1}{n}} F(s,x)| \leq 2M\|F\|_0$, for any $s\in[0,T]$, $x\in\Omega$, $n>2\omega$, and (7.3.42), by the Dominated Convergence Theorem, we obtain that for any $t\geq 0$

$$\int_0^t P_{t-s} nR(n) F_n(s,\cdot) ds \xrightarrow{\pi} \int_0^t P_{t-s} F(s,\cdot) ds \ \text{ as } n\to\infty.$$

To verify assertion (i), it remains to check that $\partial_t u_n - \mathcal{A}u_n \xrightarrow{\pi_T} F$ as $n\to\infty$. To this purpose, consider that $\partial_t u_n - \mathcal{A}u_n = nR(n)\tilde{U}_{\frac{1}{n}} F$ and now, by (7.3.41), we get

$$nR(n)F_n(t,\cdot) \xrightarrow{\pi} F(t,\cdot) \ \text{ as } n\to\infty, \quad t\geq 0. \tag{7.3.43}$$

(ii) We recall that, by Lemma 7.3.7, assertion (i), $\tilde{U}_{\frac{1}{n}} f \in \mathcal{D}_b(\Omega)$, $f\in\mathcal{C}_b(\Omega)$, $n\geq 1$ and, by Hypothesis 7.3.2, $P_t \in \mathcal{L}(\mathcal{D}_b(\Omega))$, $t\geq 0$.

Thus since $P_t R(n) = R(n)P_t$, for any $t\geq 0$, $n>\omega$, we easily find that

$$P_t nR(n) f_n \in D(\mathcal{A}_0), \quad n>\omega, \quad t\geq 0.$$

Let us consider the remainder term:

$$v_n(t,x) = \int_0^t P_{t-s} nR(n) F_n(s,x)\, ds, \quad n>\omega,\ t\in[0,T],\ x\in\Omega.$$

We fix $t>0$ and set $G_n(s,x) = P_{t-s}F_n(s,x)$, $s\in[0,t]$, $x\in\Omega$, $n\geq 1$. We claim that $G_n(\cdot,x)$ is a continuous function on $[0,t]$. To this purpose, fix $n\geq 1$ and $x\in\Omega$, by using the notations of Lemma 7.3.9 we have for any $s\in[0,t]$:

$$G_n(s,x) = P_{t-s}F_n(s,x) = \int_\Omega F_n(s,y)\, p(t-s,x,dy)$$

Since the family of measures $\{p(r,x,\cdot)\}_{r\in[0,T]}$ is tight, see (7.3.39), for any $\epsilon>0$ there exists a compact set $C_\epsilon\subset\Omega$ such that $p(r,x,\Omega\backslash C_\epsilon)<\epsilon$ for any $r\in[0,T]$. Thus we get for any $s,\ s_0 \in [0,t]$:

$$
\begin{aligned}
&|G_n(s,x) \,-\, G_n(s_0,x)| \\
&\le \Big| \int_\Omega F_n(s,y)\, p(t-s,x,dy) \,-\, \int_\Omega F_n(s_0,y)\, p(t-s_0,x,dy) \Big| \\
&\le \int_\Omega |F_n(s,y) - F_n(s_0,y)|\, p(t-s,x,dy) \\
&+ \Big| \int_\Omega F_n(s_0,y)\, p(t-s,x,dy) \,-\, \int_\Omega F_n(s_0,y)\, p(t-s_0,x,dy) \Big| \\
&\le \int_{C_\epsilon} |F_n(s,y) - F_n(s_0,y)|\, p(t-s,x,dy) \,+\, 2\epsilon \|F\|_0 \\
&+ |P_{t-s}F_n(s_0,\cdot)(x) \,-\, P_{t-s_0}F_n(s_0,\cdot)(x)|.
\end{aligned}
$$

Since F_n is uniformly continuous on $[0,T] \times C_\epsilon$, letting $s \to s_0$ in the last term of the above formula, we obtain the continuity of $G_n(\cdot, x)$.

Moreover by Hypothesis 7.3.2 and Lemma 7.3.7, we have that $G_n(s,\cdot) \in \mathcal{D}_b(\Omega)$, $s \in [0,t]$ and the following estimate holds:

$$
\begin{aligned}
&\|G_n(s,\cdot)\|_{\mathcal{D}} \,=\, \|P_{t-s}F_n(s,\cdot)\|_{\mathcal{D}} \,\le\, g(t-s)\, \|\tilde{U}_{\frac{1}{n}} F(s,\cdot)\|_{\mathcal{D}} \\
&\le\, g(t-s)\, C_n \|F(s,\cdot)\|_0 \,\le\, g(t-s)\, C_n \|F\|_0, \quad s \in [0,t],\ n \ge 1.
\end{aligned}
\tag{7.3.44}
$$

By the last estimate, applying Lemma 7.3.8, we deduce that $\int_0^t G_n(s,\cdot)ds \in \mathcal{D}_b(\Omega)$. Moreover we remark that, for any $x \in \Omega$,

$$
\int_0^t nR(n)G_n(s,x)\, ds \,=\, nR(n)\Big(\int_0^t G_n(s,x)\, ds \Big), \quad t \ge 0,\ n > \omega. \tag{7.3.45}
$$

This follows since $P_h(\int_0^t G_n(s,\cdot)ds)(x) = \int_0^t P_h G_n(s,x)ds,\ \ x \in \Omega,\ h \ge 0$ (note that $\int_0^t G_n(s,\cdot)ds$ is a π-limit of Riemann sums in $\mathcal{C}_b(\Omega)$, see (6.2.15)).

Combining the last formula with (ii) of Hypothesis 7.3.2, it follows that $v_n(t,\cdot) \in D(\mathcal{A}_0)$, $t \ge 0$, $n > \omega$. The proof is complete. ■

Let $\mathcal{B}$ be any $\pi-$closed operator on $\mathcal{C}_b(\Omega)$ and let $\mathcal{B}_0$ be a (linear) restriction of $\mathcal{B}$. We say that $\mathcal{B}$ is the π-closure of $\mathcal{B}_0$, if the following condition is satisfied:
for any $f \in D(\mathcal{B})$ there exists a sequence $(f_n) \subset D(\mathcal{B}_0)$ such that

$$
f_n \xrightarrow{\pi} f, \quad \mathcal{B}_0 f_n \xrightarrow{\pi} \mathcal{B}f \text{ as } n \to \infty. \tag{7.3.46}
$$

In the next result, we show that under the assumptions of Hypothesis 7.3.2, $\mathcal{A}$ is the π-closure of $\mathcal{A}_0$.

Proposition 7.3.10 *Let P_t be a π-semigroup on $\mathcal{C}_b(\Omega)$ with generator $\mathcal{A}$. Let $\mathcal{A}_0$ be a restriction of $\mathcal{A}$. Suppose that P_t and $\mathcal{A}_0$ verify the assumptions of Hypothesis 7.3.2 with respect to $\mathcal{D}_b(\Omega)$. Then $\mathcal{A}$ is the π-closure of $\mathcal{A}_0$.*

Proof Take any $f \in D(\mathcal{A})$ and fix $\lambda > \omega$. We set $g = (\lambda - \mathcal{A})f \in \mathcal{C}_b(\Omega)$. Denote by $\tilde{U}_{\frac{1}{n}}$ the O-U approximations on $\mathcal{C}_b(\Omega)$ and set $g_n = \tilde{U}_{\frac{1}{n}} g$, $n \geq 1$. By Lemma 7.3.7, $g_n \in \mathcal{D}_b(\Omega)$ for any $n \geq 1$ and moreover $g_n \xrightarrow{\mathcal{K}} g$ as $n \to \infty$. Define $f_n = R(\lambda, \mathcal{A}) g_n$, $n \geq 1$. By Hypothesis 7.3.2, we know that $f_n \in D(\mathcal{A}_0)$ for any $n \geq 1$. Let us notice that since

$$\lambda f_n - \mathcal{A} f_n = g_n, \quad n \geq 1,$$

If we prove that $f_n \xrightarrow{\pi} f$ as $n \to \infty$ we deduce also that $\mathcal{A}_0 f_n \xrightarrow{\pi} \mathcal{A} f$. Since

$$f_n(x) = \int_0^\infty e^{-\lambda u} P_u g_n(x)\, du, \quad x \in \Omega, \ n \geq 1,$$

and $\|P_t g_n\|_0 \leq M e^{\omega t} \|g\|_0$ for $t \geq 0$ and $n \geq 1$, by the Dominated Convergence Theorem we have $f_n \xrightarrow{\pi} f$. This completes the proof. ■

Proof of Theorem 7.3.5. Using the same notations of the proof of Theorem 7.3.4, we consider the maps, for $x \in \Omega$, $n > \omega$, $t \geq 0$,

$$u_n(t,x) = P_t nR(n) \tilde{U}_{\frac{1}{n}} f(x) + \int_0^t P_{t-s} nR(n) \tilde{U}_{\frac{1}{n}} F(s,x)\, ds, \tag{7.3.47}$$

where $\tilde{U}_t$ stands for the O-U approximations on $\mathcal{C}_b(\Omega)$, introduced in Definition 7.3.6, $R(n) = R(n, \mathcal{A})$ for $n > \omega$ and we write $\tilde{U}_{\frac{1}{n}} F(s,x) = \tilde{U}_{\frac{1}{n}} \, (F(s,\cdot))(x)$, $x \in \Omega$, $s \in [0,T]$, $n \geq 1$. In light of Theorem 7.3.4, it is enough to prove that the maps (u_n) satisfy assertion (i).
We set $f_n = \tilde{U}_{\frac{1}{n}} f$, $F_n = \tilde{U}_{\frac{1}{n}} F$, $n > \omega$. The proof will be split up into some steps.

Claim 1. $P_t nR(n) f_n \xrightarrow{\mathcal{K}_T} P_t f$ as $n \to \infty$.

Fix a compact set K in Ω. Since $\|P_t nR(n) f_n\|_0 \leq 2M e^{\omega T} \|f\|_0$, $n > 2\omega$, $t \in [0,T]$ it suffices to verify that it holds:

$$\lim_{n \to \infty} \sup_{t \in [0,T],\, x \in K} |P_t nR(n) f_n(x) - P_t f(x)| = 0. \tag{7.3.48}$$

We already know by formula (7.3.37) of the proof of Lemma 7.3.9 that $nR(n) f_n \xrightarrow{\mathcal{K}} f$ as $n \to \infty$.

Setting $g_n = nR(n) f_n$, $n > \omega$, we prove that

$$\lim_{n \to \infty} \sup_{t \in [0,T],\, x \in K} |P_t\, (g_n - f)(x)| = 0. \tag{7.3.49}$$

Denote by $p(t,x,B)$ the transition Markov function associated with P_t, with $x \in \Omega$, $t \geq 0$ and B a Borel set of Ω (see Definition 6.2.15). Since P_t satisfies condition (7.3.6), we deduce that the family of measures $\{p(t,x,\cdot)\}_{t \in [0,T],\, x \in K}$ is weakly relatively compact.

To see this fact, take any two sequences $(t_n) \subset [0,T]$ and $(x_n) \subset K$. There exist the subsequences (x_j) of (x_n) and (t_j) of (t_n) such that $x_j \to z \in K$ and $t_j \to r \in [0,T]$. We verify that $p(t_j, x_j, \cdot)$ converges weakly to $p(r,z,\cdot)$. Fix any $g \in \mathcal{C}_b(\Omega)$ and consider for any $j \geq 1$,

$$
\begin{aligned}
&|\int_\Omega g(y)p(t_j,x_j,dy) \;-\; \int_\Omega g(y)p(r,z,dy)| \\
&\leq |\int_\Omega g(y)p(t_j,x_j,dy) - \int_\Omega g(y)p(r,x_j,dy)| + |\int_\Omega g(y)p(r,x_j,dy) - \int_\Omega g(y)p(r,z,dy)| \\
&\leq \sup_{x\in K} |P_{t_j}g(x) - \; P_r g(x)| + |P_r g(x_j) - P_r g(z)|.
\end{aligned}
\tag{7.3.50}
$$

Letting $j \to \infty$ in the last term, we find the statement. Thus $p(t,x,\cdot)_{t\geq 0,\, x\in K}$ is weakly relatively compact. Applying the Prokhorov Theorem, we derive that the family of measures $\{p(t,x,\cdot)\}_{t\in[0,T],\, x\in K}$ is tight. Using this fact, we are going to prove (7.3.49).

For any $\epsilon > 0$, there exists a compact set C_ϵ in Ω such that $p(t,x,\Omega\backslash C_\epsilon) < \epsilon$ for any $t \in [0,T]$, $x \in K$. We obtain for any $n > \omega$,

$$
\begin{aligned}
&\sup_{t\in[0,T],\, x\in K} |P_t\,(g_n - f)(x)| \leq \sup_{t\in[0,T],\, x\in K} \int_\Omega |g_n(y) - f(y)| p(t\,,x,dy) \\
&\leq \sup_{t\in[0,T],\, x\in K} \Big[\int_{C_\epsilon} |g_n(y) \;-\; f(y)|\, p(t,x,dy) \;+\; \|g_n - f\|_0\, p(t,x,\Omega\backslash C_\epsilon) \Big]
\end{aligned}
$$

Taking into account that $g_n \xrightarrow{\mathcal{K}} f$ as $n \to \infty$, we get that there exists n_0 such that for any $n \geq n_0$, it holds

$$
\sup_{t\in[0,T],\, x\in K} |P_t\,(g_n - f)(x)| \leq \; \epsilon\,(1 + 3\|f\|_0) \;\text{ and claim 1 is proved .}
$$

Claim 2. $\quad u_n \xrightarrow{\mathcal{K}_T} u$ as $n \to \infty$.

In light of claim 1, it remains to check that $v_n \xrightarrow{\mathcal{K}_T} v$ as $n \to \infty$, where v_n and v are defined as follows, for any $x \in \Omega$, $n > \omega$ and $t \geq 0$,

$$
v_n(t,x) = \int_0^t P_{t-s} nR(n) F_n(s,x) ds, \quad v(t,x) = \int_0^t P_{t-s} F(s,x) ds. \tag{7.3.51}
$$

Let us consider any compact set K in Ω. We set $P_\xi = 0$ for any $\xi < 0$. Thus, for any $n > \omega$, one has

$$
\begin{aligned}
&\sup_{t\in[0,T],\, x\in K} |v_n(t,x) \;-\; v(t,x)| \leq \sup_{t\in[0,T],\, x\in K} B(n,t,x), \quad \text{where} \\
&B(n,t,x) \;=\; \int_0^T |P_{t-s} nR(n) F_n(s,x) - P_{t-s} F(s,x)| ds,\; n > \omega,\; t \in [0,T],\; x \in K.
\end{aligned}
\tag{7.3.52}
$$

We want to prove that

$$
\lim_{n\to\infty} \sup_{t\in[0,T],\, x\in K} B(n,t,x) \;=\; 0. \tag{7.3.53}
$$

Arguing by contradiction, we assume that (7.3.53) is not true. This means that there exists $\epsilon_0 > 0$ and two sequences (t_j) in $[0,T]$ and $(n_j) \subset \mathbb{N}$, such that:

$$\sup_{x\in K} \; B(n_j, t_j, x) \; > \epsilon_0, \quad j \geq 1. \tag{7.3.54}$$

There exists a subsequence of (t_j), again denoted by (t_j), that converges to some $r \in [0,T]$. Setting $n_j = j$ for convenience, in order to obtain a contradiction we will prove that $\lim_{j\to\infty} \sup_{x\in K} B(j, t_j, x) = 0$.

To this purpose consider that for any $s \in [0, T[$, $j \geq 1$, it holds

$$\begin{aligned} &\sup_{x\in K} |P_{t_j - s} jR(j)F_j(s,x) - \; P_{t_j-s}F(s,x)| \\ &\leq \sup_{w\in[0,T]} \sup_{x\in K} |P_w jR(j)F_j(s,x) - \; P_w F(s,x)|. \end{aligned} \tag{7.3.55}$$

Now for any $s \in [0, T[$, the last term of (7.3.55) tends to 0 as $j \to \infty$ by claim 1 (with f_n and f replaced respectively by $F_j(s,\cdot)$ and $F(s,\cdot)$). Moreover for j large enough

$$\sup_{x\in K} |P_{t_j-s} jR(j)F_j(s,x) - P_{t_j-s}F(s,x)| \leq 4M\, e^{\omega T} \|F\|_0, \;\; s \in [0, T[.$$

Hence we can apply the Dominated Convergence Theorem (see Lemma 7.2.3 for details) in order to obtain that $\lim_{j\to\infty} \sup_{x\in K} B(j, t_j, x) = 0$. Now (7.3.53) follows and claim 2 is proved.

Claim 3. $\partial_t u_n - \mathcal{A} u_n = nR(n)F_n \xrightarrow{\mathcal{K}_T} F$ as $n \to \infty$.

Fix a compact set K in Ω and consider for any $n > \omega$,

$$\begin{aligned} &\sup_{t\in[0,T],\, x\in K} |nR(n)F_n(t,x) \; - \; F(t,x)| \\ &\leq \sup_{t\in[0,T],\, x\in K} \int_0^\infty e^{-v}\, |P_{\frac{v}{n}} F_n(t,x) - \; F(t,x)|\, dv \\ &\leq \sup_{t\in[0,T],\, x\in K} \int_0^\infty e^{-v}\, (|P_{\frac{v}{n}} F_n(t,x) - P_{\frac{v}{n}} F(t,x)| \; + \; |P_{\frac{v}{n}} F(t,x) - F(t,x)|)\, dv. \end{aligned} \tag{7.3.56}$$

Once we have proved that that for any fixed $v > 0$,

$$\begin{aligned} &(a) \;\; \lim_{n\to\infty} \sup_{t\in[0,T],\, x\in K} |P_{\frac{v}{n}} F_n(t,x) \; - \; P_{\frac{v}{n}} F(t,x)| \; = \; 0, \\ &(b) \;\; \lim_{n\to\infty} \sup_{t\in[0,T],\, x\in K} |P_{\frac{v}{n}} F(t,x) - F(t,x)| \; = \; 0, \end{aligned} \tag{7.3.57}$$

claim 3 follows from (7.3.56) by using Lemma 7.2.3. Let us first consider statement (b). Fix $v > 0$ and assume by contradiction that (b) is not true.

Then there exists $\epsilon_0 > 0$ and a sequence (t_i) in $[0, T]$ such that:

$$\sup_{x\in K} |P_{\frac{v}{n_i}} F(t_i, x) \; - \; F(t_i, x)| \geq \epsilon_0, \;\; i \geq 1. \tag{7.3.58}$$

Now there exists a subsequence (t_j) of (t_i) such that (t_j) converges to some $r \in [0, T]$

and, setting $n_j = j$ for convenience, we can write:

$$0 < \epsilon_0 \le \sup_{x\in K} |P_{\frac{v}{j}} F(t_j, x) - F(t_j, x)| \le \Gamma_j^1 + \Gamma_j^2 + \Gamma_j^3, \quad j \ge 1,$$

$$\text{where} \quad \Gamma_j^1 = \sup_{x\in K} |P_{\frac{v}{j}} F(t_j, x) - P_{\frac{v}{j}} F(r, x)|, \tag{7.3.59}$$

$$\Gamma_j^2 = \sup_{x\in K} |P_{\frac{v}{j}} F(r, x) - F(r, x)|, \quad \Gamma_j^3 = \sup_{x\in K} |F(r, x) - F(t_j, x)|.$$

We will obtain a contradiction by showing that $\lim_{j\to\infty} \Gamma_j^1 + \Gamma_j^2 + \Gamma_j^3 = 0$.

First remark that since F is uniformly continuous on $[0, T] \times K$, it follows that $\lim_{j\to\infty} \Gamma_j^3 = 0$. Concerning Γ_j^2, by assumption (7.3.6) (with f replaced by $F(r, \cdot)$) we obtain that $\lim_{j\to\infty} \Gamma_j^2 = 0$.

It remains to consider Γ_j^1. We use that the family of measures $\{p(s, x, \cdot)\}_{s\in[0,T],\, x\in K}$ is tight, see (7.3.50). For any $\epsilon > 0$ for any $v > 0$, there exists a compact set C_ϵ in Ω such that $p(v/j, x, \Omega\backslash C_\epsilon) < \epsilon$ for any j sufficiently large and for any $x \in K$. Thus we obtain,

$$\begin{aligned}\Gamma_j^1 &\le \sup_{x\in K} \int_\Omega |F(t_j, y) - F(r, y)|\, p(v/j\,, x, dy) \\ &\le \sup_{x\in K} \int_{C_\epsilon} |F(t_j, y) - F(r, y)|\, p(v/j, x, dy) \,+\, 2\epsilon \|F\|_0\end{aligned}$$

Taking into account that F is uniformly continuous on $[0, T] \times C_\epsilon$, we obtain that for j sufficiently large, $\Gamma_j^1 \le \epsilon\, (1 + 2\|F\|_0)$. Thus condition (b) of (7.3.57) is verified. Let us consider (a) of (7.3.57). First we claim that

$$F_n = \tilde{U}_{\frac{1}{n}} F \xrightarrow{\mathcal{K}_T} F \ \text{ as } \ n \to \infty. \tag{7.3.60}$$

To see this fact, take into account that, by Lemma 7.3.7, for any $f \in \mathcal{C}_b(\Omega)$,

$$\tilde{U}_{1/n} f(x) = \int_\Omega f(y) q(1/n, x, dy), \quad x \in \Omega,\, n \ge 1,$$

where $\{q(1/n, x, \cdot)\}_{x\in\Omega,\, n\ge1}$ is a family of Borel positive measures such that: $q(1/n, x, \Omega) \le 1$, $x \in \Omega$, $n \ge 1$. Proceeding as in formula (7.3.50), using (iii) of Lemma 7.3.7, one can easily get that for any compact set K in Ω, $\{q(1/n, x, \cdot)\}_{n\ge1,\, x\in K}$ is tight. At this point arguing as for (b) of (7.3.57) we deduce that for any compact set K in Ω, it holds:

$$\lim_{n\to\infty} \sup_{t\in[0,T],\, x\in K} |\tilde{U}_{\frac{1}{n}} F(t, x) \,-\, F(t, x)| \,=\, 0$$

and (7.3.60) is verified.

Now consider again that the family of measures $\{p(s, x, \cdot)\}_{s\in[0,T],\, x\in K}$ is tight, see (7.3.50). Hence for any $\epsilon > 0$ for any $v > 0$ there exists a compact set C_ϵ in Ω such that $p(v/n, x, \Omega\backslash C_\epsilon) < \epsilon$ for n sufficiently large and for $x \in K$. We obtain

$$\begin{aligned}&\sup_{t\in[0,T],\, x\in K} |P_{\frac{v}{n}} F_n(t, x) - P_{\frac{v}{n}} F(t, x)| \\ &\le \sup_{t\in[0,T],\, x\in K} \int_{C_\epsilon} |F_n(t, y) - F(t, y)|\, p(v/n, x, dy) \,+\, 2\epsilon \|F\|_0 \\ &\le \epsilon\, (1 + 2\|F\|_0), \quad \text{for } \ n \text{ sufficiently large.}\end{aligned}$$

In the last estimate we have used formula (7.3.60). Thus also condition (a) of (7.3.57) is verified and claim 3 follows. The proof is complete. ■

Remark 7.3.11 We focus our attention on the Banach space $\mathcal{D}_b(\Omega)$. In Definition 7.3.1, for the sake of simplicity, we have stated that $\mathcal{D}_b(\Omega)$ is one of the following two spaces: $\mathcal{C}_b^1(\Omega)$ or $\tilde{\mathcal{C}}_b^2(\Omega)$.

This choice is not forced and now we show that Theorem 7.3.4 and Theorem 7.3.5 hold also if, in Hypothesis 7.3.2, $\mathcal{D}_b(\Omega)$ is a more general subspace of $\mathcal{C}_b(\Omega)$. This will be useful for applications to the Ornstein-Uhlenbeck semigroups, see §7.4.3.

In order to characterize $\mathcal{D}_b(\Omega)$, we introduce the following definition.

A linear subspace $F(\Omega)$ of $\mathcal{C}_b(\Omega)$ is called an **admissible subspace** of $\mathcal{C}_b(\Omega)$ if there exists a norm $\|\cdot\|_F$, stronger than $\|\cdot\|_0$, such that $(F(\Omega), \|\cdot\|_F)$ is a Banach space and further the following condition is satisfied:

for any map $G : [0,T] \times \Omega \to \mathbb{R}$ such that:

$$\begin{aligned}
&(i)\ \ G(\cdot, x) \text{ is a Borel map for any } x \in \Omega,\\
&(ii)\ \ G(s,\cdot) \in F(\Omega), \quad s \in [0,T],\\
&(iii)\ \ \|G(s,\cdot)\|_F \le g(s),\ \ s \in [0,T], \ \text{ where } \ g \in L^1([0,T]),
\end{aligned} \tag{7.3.61}$$

we have that the map $\phi : \Omega \to \mathbb{R}$,

$$\phi(x) = \int_0^T G(s,x)\,ds, \quad x \in \Omega \qquad \text{belongs to } \ F(\Omega).$$

Theorems 7.3.4 and 7.3.5 continue to hold, with the same proofs, if $\mathcal{D}_b(\Omega)$ is an admissible subspace of $\mathcal{C}_b(\Omega)$ and further it satisfies the following assumption:

$$\tilde{U}_t \in \mathcal{L}(\mathcal{C}_b(\Omega), \mathcal{D}_b(\Omega)), \quad t > 0, \tag{7.3.62}$$

where $\tilde{U}_t$ are the O-U approximations on $\mathcal{C}_b(\Omega)$ (see Definition 7.3.6).

For instance $\mathcal{D}_b(\Omega)$ can be $\mathcal{C}_b^2(\Omega)$ or the space consisting of all Lipschitz continuous real and bounded mappings on Ω. ▪

7.4 Some infinite dimensional parabolic problems

Here we want to apply Theorem 7.3.4 and Theorem 7.3.5 to some concrete Cauchy problems for parabolic equations with infinitely many variables. These problems involve the following transition semigroups: the heat and the Ornstein-Uhlenbeck semigroups in $\mathcal{C}_b(H)$ and the one associated with an infinite dimensional Dirichlet problem, see Chapter 5. Each Cauchy problem will be discussed in a subsection.

The main result of this section is Theorem 7.4.7, concerning the approximation of strong solutions of a Cauchy problem associated with the Ornstein-Uhlenbeck semigroup. It extends Theorem 5.8 in Cerrai and Gozzi [15].

We recall that Q stands for a self-adjoint positive and trace class operator on a real separable Hilbert space H (with inner product $<\cdot,\cdot>$ and norm $|\cdot|$) and Tr (Q) stands for the trace of Q. We fix once and for all an orthonormal basis of H, $\{e_k\}_{k\geq 1}$, that diagonalizes Q, for any $x\in H$, $Qx=\sum_{k=1}^{\infty}\lambda_k x_k e_k$ with $x_k=<x,e_k>$.

7.4.1 The Cauchy problem for the heat semigroup

We are concerned with the following initial value problem, for a fixed $T>0$,

$$\begin{cases} \partial_t u(t,x) = \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty}\lambda_k D_{kk}u(t,x)+F(t,x), & t\in]0,T],\ x\in H,\\[2ex] u(0,x)= f(x), & x\in H, \end{cases} \tag{7.4.1}$$

where $f\in\mathcal{C}_b(H)$, $F\in\mathcal{C}_\pi([0,T];\mathcal{C}_b(H))$ and $D_{kk}u$ denotes the second partial derivative of u in the direction of e_k, $k\geq 1$.

Let O_t be the heat semigroup on $\mathcal{C}_b(H)$ associated with the Gaussian measure $\mathcal{N}(0,tQ)$ and denote by $\mathcal{A}$ its generator.

Definition 7.4.1 We consider the following linear operator $\mathcal{A}_0$ on $\mathcal{C}_b(H)$, that is similar to the operator $\mathcal{A}_1$, introduced in formula (3.3.1) of Chapter 3,

$$\begin{cases} D(\mathcal{A}_0)= \{f\in\mathcal{C}^2_Q(H)\cap\mathcal{C}^1_b(H) \text{ such that } D^2_Q f(x)\in\mathcal{L}_1(H),\ x\in H,\\ \text{and}\quad D^2_Q f\in\mathcal{C}_b(H,\mathcal{L}_1(H))\};\\[2ex] \mathcal{A}_0 f(x)\stackrel{\text{def}}{=}\dfrac{1}{2}\text{Tr }[D^2_Q f(x)]=\dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty}\lambda_k D_{kk}f(x),\quad f\in D(\mathcal{A}_0),\ x\in H. \end{cases} \tag{7.4.2}$$

It follows by Theorem 3.3.2 that $\mathcal{A}_0$ is a restriction of $\mathcal{A}$. Moreover $D(\mathcal{A}_0)$ is dense in $D(\mathcal{A})$ with respect to the graph norm. This is a consequence of (ii) in Theorem 3.3.5. ∎

It turns out that O_t and $\mathcal{A}_0$ satisfy the assumptions of Hypothesis 7.3.2 with respect to $\mathcal{D}_b(H)=\mathcal{C}^1_b(H)$. Indeed we have:
(i) $O_t\in\mathcal{L}(\mathcal{C}^1_b(H))$ and $\|O_t\|_{\mathcal{L}(\mathcal{C}^1_b(H))}\leq 1$, $t\geq 0$.
Indeed, for any $f\in\mathcal{C}^1_b(H)$, $O_t f\in\mathcal{C}^1_b(H)$, $t\geq 0$ and further one has

$$<DO_t f(x),v>=\int_H <Df(x+y),v>\mathcal{N}(0,tQ)dy,\quad x,\,v\in H,\ t\geq 0); \tag{7.4.3}$$

(ii) $D(\mathcal{A}_0)\supset\bigcup_{\lambda>0}R(\lambda,\mathcal{A})\,(\mathcal{C}^1_b(H))$ (it follows by (b) of Proposition 3.3.3).

The next result follows by Theorem 7.3.4 and Theorem 7.3.5. However we provide here a self-contained proof of the theorem, in order to illustrate our method in a simple situation.

Theorem 7.4.2 *Let u be the strong solution of problem 7.4.1, that is*

$$u(t,x) \;=\; O_t f(x) \,+\, \int_0^t O_{t-s}\, F(s,x)ds, \quad t \in [0,T],\; x \in H. \tag{7.4.4}$$

There exists a sequence $(u_n) \subset \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(H)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{A}))$ such that:

(a) $u_n(t,\cdot) \in D(\mathcal{A}_0), \quad t \geq 0, \;\; n \geq 1.$
(b) $u_n \xrightarrow{\pi_T} u, \quad \partial_t u_n \,-\, \mathcal{A}_0 u_n \xrightarrow{\pi_T} F \quad as \quad n \to \infty.$

If in addition we have that F is continuous on $[0,T] \times H$, then the following stronger statement holds:
(b') $u_n \xrightarrow{\mathcal{K}_T} u, \quad \partial_t u_n \,-\, \mathcal{A}_0 u_n \xrightarrow{\mathcal{K}_T} F \quad as \quad n \to \infty.$

Proof We set $R(\lambda) = R(\lambda, \mathcal{A})$ for any $\lambda > 0$ and consider the following sequence of functions, for $x \in H$, $n \geq 1$, $t \geq 0$,

$$u_n(t,x) = O_t nR(n)\tilde{U}_{\frac{1}{n}} f(x) \,+\, \int_0^t O_{t-s} nR(n)\tilde{U}_{\frac{1}{n}} F(s,x)\, ds, \tag{7.4.5}$$

where $\tilde{U}_{\frac{1}{n}} F(s,x) \,=\, \tilde{U}_{\frac{1}{n}}\; (F(s,\cdot))(x), \;\; x \in H,\; s \in [0,T]$ and $\tilde{U}_t$ stands for the O-U approximations on $\mathcal{C}_b(H)$, introduced in Definition 7.3.6. We recall that $\tilde{U}_t$ is very regularizing Ornstein-Uhlenbeck semigroup on $\mathcal{C}_b(H)$, defined as follows

$$\tilde{U}_t f(x) \;=\; \int_H f(e^{t\tilde{B}}x + y)\mathcal{N}(0,Q_t)\, dy, \quad f \in \mathcal{C}_b(H),\; x \in H,\; t > 0.$$

The following two properties of $\tilde{U}_t$ will be used in the sequel:

$$\begin{aligned} &(1)\;\text{ for any compact set }\; K \subset H, \;\; \lim_{h\to 0} \sup_{x \in K} |\tilde{U}_{t+h} f(x) - \tilde{U}_t f(x)| = 0,\\ &\; f \in \mathcal{C}_b(H),\; t \geq 0. \\ &(2)\;\; \tilde{U}_t(\mathcal{C}_b(H)) \subset \mathcal{C}^1_b(H) \;\text{ and }\; \|\tilde{U}_t f\|_1 \leq C_t \|f\|_0, \;\; f \in \mathcal{C}_b(H),\; t > 0. \end{aligned} \tag{7.4.6}$$

(a) We set $\;f_n \,=\, \tilde{U}_{\frac{1}{n}} f \;$ and $\; F_n \,=\, \tilde{U}_{\frac{1}{n}} F, \;\; n \geq 1.$
Let us consider F_n. For any $s \in [0,T]$, $x \in H$, one has

$$F_n(s,x) \;=\; \int_H F(s, e^{\frac{1}{n}\tilde{B}}x + y)\mathcal{N}(0,Q_{\frac{1}{n}})\, dy,$$

It is clear that $F_n \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(H))$ so that we can use the same arguments of the proof of Theorem 7.2.7 in order to check that $u_n \in \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(H)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{A}))$.

Moreover by (2) of (7.4.6) we deduce that $F_n(s,\cdot) \in \mathcal{C}^1_b(H)$ for $s \in [0,T]$ and in addition the following estimate holds

$$\sup_{s\in[0,T]} \|D_x F_n(s,\cdot)\|_0 \;\leq\; \sup_{s\in[0,T]}\; c_n \|F(s,\cdot)\|_0 \;=\; c_n\|F\|_0, \;\; n \geq 1.$$

It follows that F_n is also continuous on $[0,T] \times H$, $n \geq 1$. Let us notice that by (i) and (ii) of (7.4.3), it follows that

$$O_t nR(n) f_n = nR(n) O_t f_n \in D(\mathcal{A}_0), \;\; n \geq 1, \;\; t \geq 0.$$

Let us consider the remainder term:

$$v_n(t,x) \;=\; \int_0^t nR(n) O_{t-s} F_n(s,x)\, ds, \quad n \geq 1,\; t \in [0,T],\; x \in H. \tag{7.4.7}$$

We fix $t > 0$ and introduce a map G_n,

$$G_n(s,x) = O_{t-s} F_n(s,x) \;=\; \int_H F_n(s, x + \sqrt{t-s}\, y)\, \mathcal{N}(0,Q) dy,\; n \geq 1,$$

where $s \in [0,t]$, $x \in H$. G_n is continuous on $[0,T] \times H$, thanks to the continuity of F_n and the Dominated Convergence Theorem. Moreover by (i) of (7.4.3), $G_n(s,\cdot) \in \mathcal{C}_b^1(H)$, $s \in [0,t]$, $n \geq 1$ and the following estimate holds:

$$\|G_n(s,\cdot)\|_1 \;\leq\; \|F_n(s,\cdot)\|_1 \;\leq\; c_n \|F\|_0. \tag{7.4.8}$$

Using the estimate (7.4.8) it is not difficult to check that the map $x \mapsto \int_0^t O_{t-s} G_n(s,x) ds$ belongs to $\mathcal{C}_b^1(H)$, for any $t \geq 0$ (see Lemma 7.3.8 for details).

Finally we have for any $x \in H$,

$$\int_0^t nR(n) G_n(s,x)\, ds \;=\; nR(n) \Big(\int_0^t G_n(s,x)\, ds \Big), \quad t \geq 0,\; n \geq 1,$$

and by (ii) of (7.4.3), it follows that $v_n(t,\cdot) \in D(\mathcal{A}_0)$, $t \geq 0$, $n \geq 1$. The proof of (a) is complete.

(b) We show that

$$nR(n) \tilde{U}_{\frac{1}{n}} f \;=\; nR(n) f_n \;\xrightarrow{\mathcal{K}}\; f, \;\; \text{as} \;\; n \to \infty. \tag{7.4.9}$$

Fix a compact set K in H. Since $\|nR(n) f_n\|_0 \;\leq\; \|\tilde{U}_{1/n} f\|_0 \;\leq \|f\|_0$, $n \geq 1$, to verify (7.4.9) it is enough to prove that

$$\lim_{n \to \infty} \sup_{x \in K} |nR(n) f_n(x) \;-\; f(x)| \;=\; 0. \tag{7.4.10}$$

Let $n \geq 1$. We consider

$$\begin{aligned} &\sup_{x \in K} |nR(n) f_n(x) \;-\; f(x)| \leq \\ &\sup_{x \in K} |nR(n) f_n(x) - \; nR(n) f(x)| + \; \|nR(n) f \;-\; f\|_0 = \; \Gamma^1(n) \;+\; \Gamma^2(n). \end{aligned} \tag{7.4.11}$$

Since O_t is a strongly continuous semigroup on $\mathcal{C}_b(H)$, we have that $\lim_{n\to\infty} \Gamma^2(n) = 0$. Let us consider the remainder term.

$$\begin{aligned} \Gamma^1(n) \;&\leq\; \sup_{x \in K} n \int_0^\infty e^{-nu}\, |O_u f_n(x) - O_u f(x)|\, du \\ &\leq\; \sup_{x \in K} \int_0^\infty e^{-v}\, |O_{\frac{v}{n}} f_n(x) - \; O_{\frac{v}{n}} f(x)|\, dv. \end{aligned} \tag{7.4.12}$$

Now we prove that for any $v > 0$, it holds:

$$\lim_{n\to\infty} \sup_{x\in K} |O_{v/n}(f_n - f)(x)| = 0. \tag{7.4.13}$$

Notice that $\|O_{v/n}(f_n - f)\|_0 \le 2\|f\|_0$, $n \ge 1$, $v \ge 0$. Hence once (7.4.13) is verified, we get $\lim_{n\to\infty} \Gamma^1(n) = 0$, by the Dominated Convergence Theorem, and formula (7.4.10) follows.

Let us check (7.4.13). It is easy to prove that the family of measures $\{\mathcal{N}(x,tQ)\}_{x\in K,\, t\in[0,T]}$ is tight for any $T > 0$. Now fix $v > 0$, for any $\epsilon > 0$ there exists a compact set $C_\epsilon \subset H$ such that $\mathcal{N}(x, \frac{v}{n}Q)\ (H\backslash C_\epsilon) < \epsilon$ for any $n \ge 1$, $x \in K$. Thus we have for any $n \ge 1$,

$$\begin{aligned} &\sup_{x\in K} |O_{\frac{v}{n}}(f_n - f)(x)| \\ &\le \sup_{x\in K} \int_{C_\epsilon} |f_n(y) - f(y)| \mathcal{N}(x, v/nQ)dy + 2\epsilon\|f\|_0, \end{aligned} \tag{7.4.14}$$

For f_n $\mathcal{K}$- converges to f, we can choose n_0 such that $\sup_{x\in C_\epsilon} |f_n(x) - f(x)| < \epsilon$, for any $n \ge n_0$. Thus we obtain

$$|O_{\frac{v}{n}}(f_n - f)(x)| \le \epsilon\,[1 + 2\|f\|_0], \quad n \ge n_0$$

and (7.4.13) is established. Now from (7.4.9) we derive that for any $t \ge 0$, $f \in \mathcal{C}_b(H)$,

$$O_t nR(n)\, f_n \xrightarrow{\pi} O_t f, \quad \text{as } n \to \infty. \tag{7.4.15}$$

Let us consider v_n (see (7.4.7)). Using the estimate: $|nR(n)\tilde{U}_{\frac{1}{n}}F(s,x)| \le \|F\|_0$, for any $s \in [0,T]$, $x \in H$, $n \ge 1$, and (7.4.15), by the Dominated Convergence Theorem, we obtain that for any $t \ge 0$,

$$\int_0^t O_{t-s} nR(n)F_n(s,\cdot)ds \xrightarrow{\pi} \int_0^t O_{t-s}F(s,\cdot)ds \ \text{ as } \ n \to \infty.$$

To verify assertion (b), it remains to check that $\partial_t u_n - \mathcal{A}u_n \xrightarrow{\pi_T} F$ as $n \to \infty$. To this purpose, consider that $\partial_t u_n - \mathcal{A}u_n = nR(n)F_n$. Now, by (7.4.9), we get in particular

$$nR(n)F_n(t,\cdot) \xrightarrow{\pi} F(t,\cdot) \ \text{ as } \ n \to \infty, \ \ t \ge 0. \tag{7.4.16}$$

(b') We split up this part of the proof into some steps.

Let $u_n(t,x) = O_t nR(n)f_n(x) + v_n(t,x)$, $t \in [0,T]$, $n \ge 1$, $x \in H$ (where v_n is defined in (7.4.7)). To prove that $u_n \xrightarrow{\mathcal{K}_T} u$, we verify separately that $O_t nR(n)f_n \xrightarrow{\mathcal{K}_T} O_t f$ and that $v_n \xrightarrow{\mathcal{K}_T} v$ as $n \to \infty$.

Claim 1. $O_t nR(n)f_n \xrightarrow{\mathcal{K}_T} O_t f$ as $n \to \infty$.

Fix a compact set K in H. For any $\epsilon > 0$ we can choose a compact set $C_\epsilon \subset H$ such that $\mathcal{N}(x,tQ)\ (H\backslash C_\epsilon) < \epsilon$ for any $t \in [0,T]$, $x \in K$. Thus we have for any $n \ge 1$,

$$\begin{aligned} &\sup_{x\in K, t\in[0,T]} |O_t(nR(n)f_n - f)(x)| \\ &\le \sup_{x\in K, t\in[0,T]} \int_{C_\epsilon} |nR(n)f_n(y) - f(y)| \mathcal{N}(x,tQ)dy + 2\epsilon\|f\|_0, \end{aligned} \tag{7.4.17}$$

Since $nR(n)f_n \xrightarrow{\mathcal{K}} f$ as $n \to \infty$ (by (7.4.9)), from (7.4.17) it easily follows claim 1.

Claim 2. $v_n \xrightarrow{\mathcal{K}_T} v$ as $n \to \infty$.

Fix a compact set K in H. We set $O_\xi = 0$ for any $\xi < 0$. Thus, for any $n \geq 1$, one has

$$\begin{aligned} &\sup_{t\in[0,T],\, x\in K} |v_n(t,x) \,-\, v(t,x)| \leq \sup_{t\in[0,T],\, x\in K} B(n,t,x), \quad \text{where} \\ &B(n,t,x) \,=\int_0^T |O_{t-s} nR(n)F_n(s,x) - O_{t-s}F(s,x)|ds,\; n>\omega,\; t\in[0,T],\; x\in K. \end{aligned} \tag{7.4.18}$$

We want to prove that

$$\lim_{n\to\infty} \sup_{t\in[0,T],\, x\in K} B(n,t,x) \,=\, 0. \tag{7.4.19}$$

Arguing by contradiction, we assume that (7.4.19) is not true. This means that there exists $\epsilon_0 > 0$ and two sequences (t_j) in $[0,T]$ and $(n_j) \subset \mathbb{N}$, such that:

$$\sup_{x\in K} B(n_j, t_j, x) \,>\epsilon_0, \quad j\geq 1. \tag{7.4.20}$$

There exists a subsequence of (t_j), again denoted by (t_j), that converges to some $r \in [0,T]$. Setting $n_j = j$ for convenience, in order to obtain a contradiction we will prove that $\lim_{j\to\infty} \sup_{x\in K} B(j,t_j,x) = 0$.

To this purpose consider that for any $s \in [0,T[$, $j \geq 1$, it holds

$$\begin{aligned} &\sup_{x\in K} |O_{t_j-s} jR(j)F_j(s,x) - \, O_{t_j-s}F(s,x)| \\ &\leq \sup_{w\in[0,T]} \sup_{x\in K} |O_w jR(j)F_j(s,x) - \, O_w F(s,x)|. \end{aligned} \tag{7.4.21}$$

Now for any $s \in [0,T[$, the last term of (7.4.21) tends to 0 as $j \to \infty$ by claim 1 (with f_n and f replaced respectively by $F_j(s,\cdot)$ and $F(s,\cdot)$). Moreover

$$\sup_{x\in K} |O_{t_j-s} jR(j)F_j(s,x) - O_{t_j-s}F(s,x)| \leq \, 2\,\|F\|_0,\; s\in[0,T[.$$

Hence we can apply the Dominated Convergence Theorem (see also Lemma 7.2.3) in order to obtain that $\lim_{j\to\infty}\sup_{x\in K} B(j,t_j,x) = 0$. Now (7.4.19) follows and claim 2 is proved.

Claim 3. $\partial_t u_n - \mathcal{A}u_n = nR(n)F_n \xrightarrow{\mathcal{K}_T} F$ as $n \to \infty$.

Fix a compact set K in H and consider for any $n \geq 1$,

$$\begin{aligned} &\sup_{t\in[0,T],\, x\in K} |nR(n)F_n(t,x) \,-\, F(t,x)| \\ &\leq \sup_{t\in[0,T],\, x\in K} \int_0^\infty e^{-v}\, |O_{\frac{v}{n}}F_n(t,x) - \, F(t,x)|\, dv \\ &\leq \sup_{t\in[0,T],\, x\in K} \int_0^\infty e^{-v}\left(|O_{\frac{v}{n}}F_n(t,x) - O_{\frac{v}{n}}F(t,x)| \,+\, |O_{\frac{v}{n}}F(t,x) - F(t,x)|\right) dv. \end{aligned} \tag{7.4.22}$$

Once we have proved that, for any fixed $v > 0$,

$$\begin{aligned} &(i)\ \lim_{n\to\infty} \sup_{t\in[0,T],\, x\in K} |O_{\frac{v}{n}} F_n(t,x) - O_{\frac{v}{n}} F(t,x)| = 0, \\ &(ii)\ \lim_{n\to\infty} \sup_{t\in[0,T],\, x\in K} |O_{\frac{v}{n}} F(t,x) - F(t,x)| = 0, \end{aligned} \tag{7.4.23}$$

using the estimate (7.4.22), we obtain claim 3 (indeed letting $n \to \infty$ in (7.4.22), by Lemma 7.2.3, we find the assertion). The proofs of assertions (i) and (ii) are based on the following fact:

$$\lim_{n\to\infty} \sup_{t\in[0,T],\, x\in K} |F_n(t,x) - F(t,x)| = 0, \tag{7.4.24}$$

that now we verify. Assume, by contradiction, that (7.4.24) is not true. This means that there exists $\epsilon_0 > 0$ and a sequence (t_m) in $[0,T]$ such that:

$$\sup_{x\in K} |\tilde{U}_{\frac{1}{n_m}} F(t_m,x) - F(t_m,x)| \geq \epsilon_0, \quad n \geq 1. \tag{7.4.25}$$

There exists a subsequence (t_j) of (t_m) such that $t_j \to r \in [0,T]$ as $j \to \infty$ and we can write, setting $j = n_{m_j}$ for convenience,

$$\begin{aligned} &0 < \epsilon_0 \leq \sup_{x\in K} |\tilde{U}_{\frac{1}{j}} F(t_j,x) - F(t_j,x)| \leq \Gamma_j^1 + \Gamma_j^2 + \Gamma_j^3, \quad j \geq 1 \\ &\text{where} \quad \Gamma_j^1 = \sup_{x\in K} |\tilde{U}_{\frac{1}{j}} F(t_j,x) - \tilde{U}_{\frac{1}{j}} F(r,x)|, \\ &\Gamma_j^2 = \sup_{x\in K} |\tilde{U}_{\frac{1}{j}} F(r,x) - F(r,x)|, \quad \Gamma_j^3 = \sup_{x\in K} |F(r,x) - F(t_j,x)|. \end{aligned} \tag{7.4.26}$$

Now we will obtain a contradiction by showing that $\lim_{j\to\infty} \Gamma_j^1 + \Gamma_j^2 + \Gamma_j^3 = 0$.

First remark that since F is uniformly continuous on $[0,T] \times K$, it follows that $\lim_{j\to\infty} \Gamma_j^3 = 0$. Concerning Γ_j^2, by (1) of (7.4.6) (with f replaced by $F(r,\cdot)$), we obtain that $\lim_{j\to\infty} \Gamma_j^2 = 0$.

It remains to consider Γ_j^1. We set for short $p(t,x,\cdot) = \mathcal{N}(e^{t\tilde{B}}x, Q_t)$, $t \geq 0$, $x \in H$. Using that the family of measures $\{p(s,x,\cdot)\}_{s\in[0,T],\, x\in K}$ is tight, for any $\epsilon > 0$ there exists a compact set C_ϵ in H such that $p(1/j, x, H \backslash C_\epsilon) < \epsilon$ for j large enough, $x \in K$. Thus we obtain

$$\begin{aligned} \Gamma_j^1 &\leq \sup_{x\in K} \int_H |F(t_j,y) - F(r,y)|\, p(1/j\,, x, dy) \\ &\leq \sup_{x\in K} \int_{C_\epsilon} |F(t_j,y) - F(r,y)|\, p(1/j, x, dy) + 2\epsilon \|F\|_0 \end{aligned}$$

Taking into account that F is uniformly continuous on $[0,T] \times C_\epsilon$, we obtain that for j sufficiently large, $\Gamma_j^1 \leq \epsilon\,(1 + 2\|F\|_0)$. Thus formula (7.4.24) is verified.

To prove condition (ii) of (7.4.23), we can proceed as for (7.4.24). It remains to check condition (i).

Fix $v > 0$ and consider that the family of measures $\{\mathcal{N}(x, \frac{v}{n}Q)\}_{n\geq 1,\, x\in K}$ is tight. Hence for any $\epsilon > 0$ there exists a compact set C_ϵ in H such that $\mathcal{N}(x, \frac{v}{n}Q)(H\backslash C_\epsilon) < \epsilon$ for any $n \geq 1$, $x \in K$. We obtain for any $n \geq 1$,

$$\begin{aligned}
&\sup_{t\in[0,T],\, x\in K} |O_{\frac{v}{n}}F_n(t,x) - O_{\frac{v}{n}}F(t,x)| \\
&\leq \sup_{t\in[0,T],\, x\in K} \int_{C_\epsilon} |F_n(t,y) - F(t,y)|\, \mathcal{N}(x, \frac{v}{n}Q)dy \,+\, 2\epsilon\|F\|_0 \\
&\leq \epsilon\,(1 + 2\|F\|_0), \quad \text{for} \quad n \text{ sufficiently large.}
\end{aligned}$$

In the last estimate, we have used formula (7.4.24). Thus also condition (i) of (7.4.23) is proved and claim 3 holds. The proof is complete. ■

7.4.2 A homogeneous Dirichlet problem in a half Space of H

We consider the following open half space of H : $H_+ = \{x \in H$ such that $x_1 =< x, e_1 > > 0\}$.

Let $\partial H_+ = \{x \in H$ such that $x_1 = 0\}$ and Q be the operator introduced at the beginning of the section. We are dealing with the following initial value problem:

$$\begin{cases} \partial_t u(t,x) \,=\, \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty} \lambda_k D_{kk}u(t,x) + F(t,x), \quad t \in]0,T],\ x \in H_+, \\ \\ u(t,z) \,=\, 0, \quad z \in \partial H_+, \quad t > 0, \\ u(0,x) = \, f(x), \qquad x \in H_+, \end{cases} \tag{7.4.27}$$

where $f \in \mathcal{C}_b(H_+)$, $F \in \mathcal{C}_\pi([0,T]; \mathcal{C}_b(H_+))$.

We set $H_+ =]0, \infty[\, \times H'$, where H' is the Hilbert space spanned by $\{e_k\}_{k\geq 2}$. We define the operator $Q' : H' \to H'$, $Q'x = \sum_{k=2}^{\infty} \lambda_k x_k e_k$, $x \in H'$. Problem (7.4.27) is naturally associated with a transition π-semigroup P_t (see Chapter 5 and §6.3.2 for more details) defined as follows:

$$P_t f(x) \,=\, \int_{\mathbb{R}_+} \left(\frac{(e^{-\frac{(x_1-y_1)^2}{2t\lambda_1}} - e^{-\frac{(x_1+y_1)^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}} \right) dy_1 \int_{H'} f(y_1, x' + y')\mathcal{N}(0', tQ')dy', \tag{7.4.28}$$

where $f \in \mathcal{C}_b(H_+)$, $x \in H$, $t > 0$. Remark that for any $f \in \mathcal{C}_b(H_+)$, we have that $P_t(z) = 0$, $z \in \partial H_+$, $t > 0$. Denote by $\mathcal{T}$ the generator of P_t.

Definition 7.4.3 We introduce the following linear operator:

$$\begin{cases} D(\mathcal{T}_0) \,=\, \{f \in \mathcal{C}_b^1(H_+) \cap \mathcal{C}_Q^2(H_+) \text{ such that } f(z) = 0,\ z \in \partial H_+ \text{ and} \\ D_Q^2 f \in \mathcal{C}_b(H_+, \mathcal{L}_1(H))\}; \\ \\ \mathcal{T}_0 \,: D(\mathcal{T}_0) \,\to\, \mathcal{C}_b(H_+), \quad \mathcal{T}_0 f(x) \overset{\text{def}}{=} \dfrac{1}{2}\text{Tr}\,[D_Q^2 f(x)],\ f \in D(\mathcal{T}_0),\ x \in H_+, \end{cases} \tag{7.4.29}$$

where the space $\mathcal{C}_Q^2(H_+)$ is defined in Section 1.3. In the same way as (i) of Theorem 5.2.13, one prove that $\mathcal{T}_0$ is a restriction of $\mathcal{T}$. Moreover the fact that $D(\mathcal{T}_0)$ is dense in $D(\mathcal{T})$ with respect to the graph norm follows by (ii) in Theorem 5.2.13. ∎

We will apply Theorem 7.3.4 and Theorem 7.3.5 by choosing $\mathcal{D}_b(H_+) = \mathcal{C}_b^1(H_+)$. To this purpose, in the next lemma, we check that P_t and $\mathcal{T}_0$ satisfy the assumptions of Hypothesis 7.3.2 with respect to $\mathcal{C}_b^1(H_+)$.

Lemma 7.4.4 *The following statements hold:*
(i) $P_t \in \mathcal{L}(\mathcal{C}_b^1(H_+))$ and $\|P_t\|_{\mathcal{L}(\mathcal{C}_b^1(H_+))} \le \frac{c}{\sqrt{t}}$, $t > 0$.
(ii) $D(\mathcal{T}_0) \supset \bigcup_{\lambda>0} R(\lambda, \mathcal{T})\,(\mathcal{C}_b^1(H_+))$.

Proof (i) For any $f \in \mathcal{C}_b^1(H_+)$, we define $D_{x'}f(x) = \sum_{k\ge 2} D_k f(x) e_k$, $x \in H_+$, where D_k denote the partial derivative with respect to e_k. Further for any $v \in H$, we set $v' = \sum_{k=2}^{\infty} v_k e_k$.

It is not difficult to verify that for a fixed $f \in \mathcal{C}_b^1(H_+)$, $P_t f$ is Fréchet differentiable on H_+ for any $t \ge 0$. Moreover the Fréchet derivative $DP_t f$ can be written as follows, for any $v \in H$, $x \in H_+$, $t \ge 0$,

$$< DP_t f(x), v > = < D_{x'} P_t f(x), v' >_{H'} + v_1 D_1 P_t f(x)$$

$$= \int_{\mathbb{R}_+} \left(\frac{(e^{-\frac{(x_1-y_1)^2}{2t\lambda_1}} - e^{-\frac{(x_1+y_1)^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}} \right) dy_1 \int_{H'} < D_{x'} f(y_1, x'+y'), v' > \mathcal{N}(0', tQ') dy'$$

$$+ \frac{2v_1}{\sqrt{2\pi t\lambda_1}} e^{-x_1^2/2t\lambda_1} \int_{H'} f(0, x'+y')\, \mathcal{N}(0', tQ') dy'$$

$$+ \int_{\mathbb{R}_+} \left(\frac{(e^{-\frac{(x_1-y_1)^2}{2t\lambda_1}} + e^{-\frac{(x_1+y_1)^2}{2t\lambda_1}}}{\sqrt{2\pi t\lambda_1}} \right) dy_1 \int_{H'} v_1 D_1 f(y_1, x'+y') \mathcal{N}(0', tQ') dy.'$$

From this equality it follows that $P_t f \in \mathcal{C}_b^1(H_+)$ and further that

$$\|DP_t f\|_0 \le \frac{c}{\sqrt{t}} \|f\|_1, \quad t > 0.$$

Hence condition (i) holds.

(ii) This assertion can be obtained following the proof of Theorem 5.2.11. ∎

Taking into account Proposition 6.3.1 and Lemma 7.4.4, from Theorem 7.3.4 and Theorem 7.3.5 (with $\mathcal{D}_b(H_+) = \mathcal{C}_b^1(H_+)$) we state the following result.

Theorem 7.4.5 *Let u be the strong solution of problem 7.4.27, that is*

$$u(t,x) = P_t f(x) + \int_0^t P_{t-s} F(s,x) ds, \quad t \in [0,T],\ x \in H_+. \tag{7.4.30}$$

There exists a sequence $(u_n) \subset \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(H_+)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{T}))$ *such that:*

(a) $u_n(t,\cdot) \in D(\mathcal{T}_0), \quad t \geq 0, \quad n \geq 1.$
(b) $u_n \xrightarrow{\pi_T} u, \quad \partial_t u_n - \mathcal{T}_0 u_n \xrightarrow{\pi_T} F \quad as \quad n \to \infty.$

If in addition we have that F *is continuous on* $[0,T] \times H_+$*, then the following stronger statement holds:*
(b') $u_n \xrightarrow{\mathcal{K}_T} u, \quad \partial_t u_n - \mathcal{T}_0 u_n \xrightarrow{\mathcal{K}_T} F \quad as \quad n \to \infty.$

7.4.3 The Cauchy problem for the Ornstein-Uhlenbeck semigroup

Here we follow the notations of Section 6.3.3. Let M be a self-adjoint, non negative, bounded linear operator on H. Let A be the generator of a strongly continuous semigroup S_t on H. We suppose that there exists $\omega < 0$ and $C > 0$ such that $\|S_t\|_{\mathcal{L}(H)} \leq C\, e^{\omega t}, \;\; t \geq 0$. This assumption is not restrictive. Indeed by standards arguments, it is possible to adapt all the proofs to the general case of $\omega \geq 0$.

In addition we assume that for each $t \geq 0$, the bounded linear operators $M(t)$,

$$M(t)x \;=\; \int_0^t S_u M S_u^* \, x \, du, \quad x \in H,$$

belong to $\mathcal{L}_1(H)$. Here S_t^* denotes the adjoint semigroup of S_t. We are dealing with the following initial value problem:

$$\begin{cases} \partial_t u(t,x) = \dfrac{1}{2}\mathrm{Tr}\,[MD^2u(t,x)] + < Ax, Du(t,x) > + F(t,x), \;\; x \in D(A),\, t \in]0,T] \\ \\ u(0,x) \;=\; f(x), \qquad x \in H, \end{cases} \tag{7.4.31}$$

where $f \in \mathcal{C}_b(H)$ and $F \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(H))$. We denote by U_t the Ornstein - Uhlenbeck semigroup on $\mathcal{C}_b(H)$, related to M and S_t, and by $\mathcal{U}$ its generator. Under our assumptions one has

$$U_t f(x) \;=\; \int_H f(S_t x + y)\, \mathcal{N}(0, M(t))\, dy, \quad f \in \mathcal{C}_b(H),\; x \in H,\; t > 0, \tag{7.4.32}$$

Definition 7.4.6 We define a natural restriction of $\mathcal{U}$. To this purpose we need some notations. For any $f \in \mathcal{C}_b(H)$, following Ahmed et al[1, §2.3], we define the map

$$f_A \,:\, D(A) \subset H \;\to \mathbb{R}, \;\; f_A(x) = \; f \circ A(x) \;= f(Ax), \; x \in D(A). \tag{7.4.33}$$

We shall write $f_A \in \mathcal{C}_b(H)$ if f_A has a uniformly continuous extension to the whole of H. This extension, that is unique, will be again denoted by f_A.

Notice that if $f \in \mathcal{C}^1_b(H)$ and in addition $f_A \in \mathcal{C}^1_b(H)$, then

$$Df(x) \in D(A^*), \;\; x \in H \;\; \text{and} \quad A^*Df \in \mathcal{C}_b(H,H). \tag{7.4.34}$$

To see this fact, set $f_A = g$, then $g(A^{-1}y) = f(y)$, $y \in H$ (take into account that $A^{-1} \in \mathcal{L}(H)$). Hence we can write: $< Df(y), v > = < Dg(A^{-1}y), A^{-1}v >$, $y, v \in H$. Setting $v = Aw$ with $w \in D(A)$ one has: $< Df(y), Aw > = < Dg(A^{-1}y), w >$. Thus $A^*Df(y) = Dg(A^{-1}y)$, $y \in H$ and formula (7.4.34) is verified.

We recall the Banach space $(\tilde{\mathcal{C}}_b^2(H), \|\cdot\|_{\tilde{2}})$, introduced in Definition 7.3.1:

$$\tilde{\mathcal{C}}_b^2(H) \stackrel{\text{def}}{=} \{ f \in \mathcal{C}_b^2(H), \;\; \text{such that} \;\; D^2 f \in \mathcal{C}_b(H, \mathcal{L}_1(H)) \},$$

$\|f\|_{\tilde{2}} = \|f\|_2 + \sup_{x\in H} \|D^2 f(x)\|_{\mathcal{L}_1(H)}$, $f \in \tilde{\mathcal{C}}_b^2(H)$.

Now we consider the following linear operator $\mathcal{U}_0 : D(\mathcal{U}_0) \subset \mathcal{C}_b(H) \to \mathcal{C}_b(H)$,

$$\begin{cases} D(\mathcal{U}_0) = \{ f \in \tilde{\mathcal{C}}_b^2(H) \text{ such that } \; f_A \in \mathcal{C}_b^1(H) \\ \text{and the map } \; x \mapsto < A^*Df(x), x > \text{ belongs to} \, \mathcal{C}_b(H) \}. \\ \\ \mathcal{U}_0 f(x) \stackrel{\text{def}}{=} \dfrac{1}{2} \text{Tr} \, [M D^2 f(x)] + \; < A^*Df(x), x >, \;\; f \in D(\mathcal{U}_0), \; x \in H, \end{cases} \tag{7.4.35}$$

where $D^2 f$ stands for the second Fréchet derivative of f. $\mathcal{U}_0$ was introduced in Cerrai and Gozzi [15, §5.7], where it was also proved that $\mathcal{U}_0$ is a restriction of $\mathcal{U}$. ∎

The main result of this section is the following theorem that improves Theorem 5.8 in Cerrai and Gozzi [15].

Theorem 7.4.7 *Let u be the strong solution of problem 7.4.31, namely*

$$u(t,x) \;=\; U_t f(x) \;+\; \int_0^t U_{t-s} \, F(s,x) ds, \quad t \in [0,T], \; x \in H. \tag{7.4.36}$$

There exists a sequence $(u_n) \subset \mathcal{C}_\pi^1([0,T]; \mathcal{C}_b(H)) \cap \mathcal{C}_\pi([0,T]; D(\mathcal{U}))$ such that:

(a) $u_n(t,\cdot) \in D(\mathcal{U}_0), \quad t \geq 0, \;\; n \geq 1.$
(b) $u_n \xrightarrow{\pi_T} u, \quad \partial_t u_n - \mathcal{U}_0 u_n \;\xrightarrow{\pi_T}\; F$ *as* $n \to \infty$.

If in addition we have that F is continuous on $[0,T] \times H$, then the following stronger statement holds:
(b') $u_n \xrightarrow{\mathcal{K}_T} u, \quad \partial_t u_n - \mathcal{U}_0 u_n \;\xrightarrow{\mathcal{K}_T}\; F$ *as* $n \to \infty$.

For the proof we need the following functions space:

$$\tilde{\mathcal{C}}_A^2(H) \stackrel{\text{def}}{=} \{ f \in \tilde{\mathcal{C}}_b^2(H), \;\; \text{such that} \quad f_A \in \tilde{\mathcal{C}}_b^2(H) \}. \tag{7.4.37}$$

This space was introduced in Ahmed et al [1]. It is easy to prove that $\tilde{\mathcal{C}}_A^2(H)$ is a Banach space, endowed with the norm:

$$\|f\|_{\tilde{2},A} \;=\; \|f\|_{\tilde{2}} \;+\; \|f_A\|_{\tilde{2}}, \quad f \in \tilde{\mathcal{C}}_A^2(H).$$

In order to prove Theorem 7.4.7, we will show that Theorems 7.3.4 and 7.3.5 hold for U_t and $\mathcal{U}_0$, when the space $\mathcal{D}_b(H)$ is replaced by $\tilde{\mathcal{C}}^2_A(H)$.

This goal will be achieved according to the following plan.
Step 1. We verify that $\tilde{\mathcal{C}}^2_A(H)$ is *an admissible subspace* of $\mathcal{C}_b(\Omega)$, see Remark 7.3.11.
Step 2. We show that U_t and $D(\mathcal{U}_0)$ satisfy the assumptions of Hypothesis 7.3.2 when $\mathcal{D}_b(H)$ is replaced by $\tilde{\mathcal{C}}^2_A(H)$, see Lemma 7.4.8.
Step 3. We remark that one can not apply directly the proofs of Theorems 7.3.4 and 7.3.5, with $\mathcal{D}_b(H)$ replaced by $\tilde{\mathcal{C}}^2_A(H)$. Indeed it is not clear if $\tilde{\mathcal{C}}^2_A(H)$ fulfils (7.3.62) of Remark 7.3.11. However we can overcome this difficulty by means of Lemmas 7.4.9 and 7.4.10. These results allow to adapt, with few changes, the proofs of Theorems 7.3.4 and 7.3.5 in order to obtain Theorem 7.4.7.

We obtain easily step 1. Indeed to see that $\tilde{\mathcal{C}}^2_A(H)$ is an admissible subspace, we simply apply Lemma 7.3.8, taking into account that if $G : [0,T] \times H \to \mathbb{R}$ is a map such that $G(s,\cdot) \in \tilde{\mathcal{C}}^2_A(H)$ for $s \in [0,T]$ and $G(\cdot,x)$ is Borel for $x \in H$, then the map W,

$$W(s,x) \stackrel{\text{def}}{=} G(s,Ax), \quad s \in [0,T], \ x \in D(A), \tag{7.4.38}$$

is Borel in s and such that $W(s,\cdot) \in \tilde{\mathcal{C}}^2_b(H)$ for $s \in [0,T]$.

Step 2 is obtained by the following result.

Lemma 7.4.8 *The following statements hold:*

$$\begin{aligned} &(i) \ \ U_t \in \mathcal{L}(\tilde{\mathcal{C}}^2_A(H)) \ \text{ and } \|U_t f\|_{\tilde{2},A} \le \|f\|_{\tilde{2},A}, \ t \ge 0, \ f \in \tilde{\mathcal{C}}^2_A(H); \\ &(ii) \ \ D(\mathcal{U}_0) \supset \bigcup_{\lambda > 0} R(\lambda, \mathcal{U})\,(\tilde{\mathcal{C}}^2_A(H)). \end{aligned} \tag{7.4.39}$$

Proof **(i)** Let $f \in \tilde{\mathcal{C}}^2_A(H)$ and $t > 0$, we prove that $U_t f \in \tilde{\mathcal{C}}^2_A(H)$.

It is not difficult to verify that $U_t f \in \tilde{\mathcal{C}}^2_b(H)$. We only remark that for the second Fréchet derivative of U_t one has:

$$D^2 U_t f(x) \,=\, \int_H S_t^* \, D^2 f(S_t x + y)\, S_t \ \ \mathcal{N}(0, M(t))\, dy,$$

where the integral is in the Bochner sense with values in $\mathcal{L}_1(H)$, since $D^2 f \in \mathcal{C}_b(H, \mathcal{L}_1(H))$. This way it is clear that $D^2 U_t f \in \mathcal{C}_b(H, \mathcal{L}_1(H))$ and further that it holds: $\|U_t f\|_{\tilde{2}} \le \|f\|_{\tilde{2}}, \ t \ge 0$.
Now we deal with the map $(U_t f)_A : D(A) \subset H \to H$. For any $x \in D(A)$, one has by changing variable,

$$\begin{aligned} [U_t f](Ax) \,&=\, \int_H f(S_t A x \,+ y)\, \mathcal{N}(0, M(t))\, dy \\ &=\, \int_H f(A\,[S_t x \,+\, A^{-1} y]) \ \ \mathcal{N}(0, M(t))\, dy, \\ &=\, \int_H f_A(S_t x + z) \ \ \mathcal{N}(0, A^{-1}\, M(t)\, (A^{-1})^*)\, dz. \end{aligned} \tag{7.4.40}$$

Arguing as for $U_t f$ we obtain that $(U_t f)_A \in \tilde{\mathcal{C}}_b^2(H)$ and $\|(U_t f)_A\|_{\tilde{2}} \le \|f_A\|_{\tilde{2}}$, $t \ge 0$. The assertion (i) is proved.

(ii) Fix $\hat{f} \in \tilde{\mathcal{C}}_A^2(H)$ and $\lambda > 0$, we have to prove that

$$\psi = R(\lambda, \mathcal{U})\hat{f} = R(\lambda)\hat{f} \in D(\mathcal{U}_0).$$

By assertion (i), proceeding as in Lemma 7.3.8, it is straightforward to verify that $\psi \in \tilde{\mathcal{C}}_b^2(H)$ and $\psi_A \in \mathcal{C}_b^1(H)$.

For any $g \in \tilde{\mathcal{C}}_b^2(H)$ such that $g_A \in \mathcal{C}_b^1(H)$, we define the maps $T_1 g$ and $T_2 g$ as follows,

$$T_1 g(x) = \frac{1}{2} \operatorname{Tr}\,(M D^2 g(x)), \qquad T_2 g(x) = < x, A^* D g(x) >, \quad x \in H. \tag{7.4.41}$$

It is clear that $T_1 g \in \mathcal{C}_b(H)$ and further that $T_2 g$ is continuous on H. In order to prove that $\psi \in D(\mathcal{U}_0)$, it remains to check that $T_2 \psi \in \mathcal{C}_b(H)$.

We use the following formula that can be proved as in Da Prato and Zabczyk [23, chapter 9] (see also Cerrai and Gozzi [15, Lemma 5.6]).

$$\frac{d}{dt} U_t \hat{f}(x) = \begin{cases} T_1 U_t \hat{f}(x) + T_2 U_t \hat{f}(x), & t > 0,\ x \in H \\ \\ T_1 \hat{f}(x) + T_2 \hat{f}(x), & t = 0,\ x \in H. \end{cases} \tag{7.4.42}$$

Now we argue similarly to Zambotti [86, §5.1]. First it is simple to verify that

$$T_1 \psi(x) = \int_0^\infty e^{-\lambda t}\, T_1 U_t \hat{f}(x)\, dt, \ x \in H$$

(we only remark that Tr $(MD^2\psi)$ is independent of the choice of the orthonormal basis (e_k) and further there results, by standard arguments, for any $x \in H$, $n \ge 1$,

$$\Big| \sum_{k=1}^n < M D^2 U_t \hat{f}(x)(e_k), e_k > \Big| \le \|M\|_{\mathcal{L}(H)}\, \|\hat{f}\|_{\tilde{2}} \,).$$

Then, integrating by parts and using (7.4.42), we get for any $x \in H$,

$$T_1\psi(x) + T_2\psi(x) = \int_0^\infty e^{-\lambda t}\, (T_1 U_t \hat{f}(x) + T_2 U_t \hat{f}(x)) dt$$

$$= \int_0^\infty e^{-\lambda t}\, \frac{d}{dt} U_t \hat{f}(x)\, dt = -\hat{f}(x) + \lambda \int_0^\infty e^{-\lambda t} U_t \hat{f}(x)\, dt$$

$$= -\hat{f}(x) + \lambda \psi(x).$$

It follows that $T_2\psi = -T_1\psi - \hat{f} + \lambda\psi$ and so $T_2\psi \in \mathcal{C}_b(H)$. Hence $\psi \in D(\mathcal{U}_0)$ and the proof is complete. ∎

We prepare the proof of Theorem 7.4.7 with two preliminary results, see step 3.

Let us recall that $\tilde{U}_t$ denotes the O-U approximations on $\mathcal{C}_b(H)$ (that coincides with the semigroup $\tilde{Z}_t$, see Definition 7.3.6 for details).

Lemma 7.4.9 *Set* $L_n = nR(n,A)$, $n \geq 1$. *One has:*

(i) *for any* $g \in \tilde{\mathcal{C}}_b^2(H)$, $g \circ L_n \in \tilde{\mathcal{C}}_A^2(H)$, $n \geq 1$ *and further* $g \circ L_n \xrightarrow{\mathcal{K}} g$ *as* $n \to \infty$;

(ii) *for any* $f \in \mathcal{C}_b(H)$, $(\tilde{U}_{\frac{1}{n}} f) \circ L_n \in \tilde{\mathcal{C}}_A^2(H)$, $n \geq 1$ *and further* $(\tilde{U}_{\frac{1}{n}} f) \circ L_n \xrightarrow{\mathcal{K}} f$ *as* $n \to \infty$.

Proof (i) Fix $g \in \tilde{\mathcal{C}}_b^2(H)$ and $n \geq 1$. We set $\phi(x) = g \circ L_n(x) = g(L_n x)$, $x \in H$. It is clear that $\phi \in \tilde{\mathcal{C}}_b^2(H)$, let us verify that also $\phi_A \in \tilde{\mathcal{C}}_b^2(H)$.

We have for any $x \in D(A)$,

$$\phi_A(x) = \phi(Ax) = g(L_n Ax) = g(AL_n x).$$

Since $AL_n \in \mathcal{L}(H)$, the map ϕ_A can be extended to a uniformly continuous map defined on the whole of H. Moreover it is straightforward to verify that $\phi_A \in \tilde{\mathcal{C}}_b^2(H)$ and its second Fréchet derivative is given by

$D^2\phi_A(x) = (AL_n)^* \, D^2 g(AL_n x) \, AL_n$, $x \in H$.

Now we prove the second part of (i). To this purpose we fix a compact set K in H and prove the following statement:

$$\lim_{n\to\infty} \sup_{x \in K} |L_n x - x| = 0. \tag{7.4.43}$$

We have for any $n \geq 1$,

$$\begin{aligned} \sup_{x\in K} |L_n x - x| &\leq \sup_{x \in K} n \int_0^\infty e^{-nu} \, |S_u x - x| \, du \\ &\leq \sup_{x\in K} \int_0^\infty e^{-v} \, |S_{\frac{v}{n}} x - x| \, dv. \end{aligned} \tag{7.4.44}$$

Now let us notice that there exists a constant C such that $x \in K$ implies that $|x| \leq C$. Therefore we have $|S_{\frac{v}{n}} x - x| \leq 2C$ for any $x \in K$, $v \geq 0$, $n \geq 1$.

Moreover S_t is a $\mathcal{C}_0$ semigroup on H, hence it is easy to prove that

$$\lim_{n\to\infty} \sup_{x\in K} |S_{\frac{v}{n}} x - x| = 0, \quad v \geq 0.$$

Letting $n \to \infty$ in the last term of (7.4.44) we obtain (7.4.43), by the Dominated Convergence Theorem (see Lemma 7.2.3 for more details).

Now denote by ω_g the modulus of continuity of g. One has for any $n \geq 1$,

$$\sup_{x\in K} \|g(L_n x) - g(x)\| \leq \sup_{x\in K} \omega_g(|L_n x - x|).$$

Letting $n \to \infty$ we find 0, by formula (7.4.43). Thus assertion (i) is proved.

(ii) Let $f \in \mathcal{C}_b(H)$, we set $f_n = (\tilde{U}_{\frac{1}{n}} f) \circ L_n$, $n \geq 1$.

By Lemma 7.3.7, we know that $\tilde{U}_{\frac{1}{n}} f \in \tilde{\mathcal{C}}_b^2(H)$ for any $n \geq 1$. In virtue of assertion (i) we can deduce that $f_n \in \tilde{\mathcal{C}}_A^2(H)$ for $n \geq 1$. We prove the second part of (ii).

First let us consider that $\|(\tilde{U}_{\frac{1}{n}} f) \circ L_n\|_0 \leq \|f\|_0$ for $n \geq 1$.

Fix a compact set K in H and $\epsilon > 0$. By the uniform continuity of f, we can find $\delta > 0$ such that for any $x, z \in H$, $\|x - z\| \le \delta$ implies that $|f(x) - f(z)| \le \epsilon$. Now arguing as in (7.3.16), we get:

$$\begin{aligned}
&\sup_{x\in K} |\tilde{U}_{1/n} f(L_n x) - f(x)| \le \sup_{x\in K} \int_H |f(e^{\frac{1}{n}\tilde{B}} L_n x + y) - f(x)| \mathcal{N}(0, Q_{\frac{1}{n}}) dy \\
&\le \sup_{x\in K} \int_{|y|<\frac{\delta}{2}} |f(e^{\frac{1}{n}\tilde{B}} L_n x + y) - f(x)| \mathcal{N}(0, Q_{1/n})\, dy \\
&+ \frac{4}{\delta} \|f\|_0 \sqrt{\mathrm{Tr}\ Q_{\frac{1}{n}}}.
\end{aligned} \tag{7.4.45}$$

Now consider the following estimate, for any $n \ge 1$:

$$\begin{aligned}
&\sup_{x\in K} |e^{\frac{1}{n}\tilde{B}} L_n x - x| \le \sup_{x\in K} (|e^{\frac{1}{n}\tilde{B}} L_n x - e^{\frac{1}{n}\tilde{B}} x| + |e^{\frac{1}{n}\tilde{B}} x - x|) \\
&\le \sup_{x\in K} (|L_n x - x| + |e^{\frac{1}{n}\tilde{B}} x - x|).
\end{aligned} \tag{7.4.46}$$

Letting $n \to 0^+$ in the last term of (7.4.46) we find, by formula (7.4.44), $\lim_{n\to\infty} \sup_{x\in K} |e^{\frac{1}{n}\tilde{B}} L_n x - x| = 0$. Hence there exists n_0 such that for any $n \ge n_0$ we have

$$|e^{\frac{1}{n}\tilde{B}} L_n x - x + y| \le \frac{\delta}{2} + |y|, \quad x \in K,\ y \in H.$$

Using this fact in (7.4.45), we obtain that for $n \ge n_0$,

$$\sup_{x\in K} |\tilde{U}_{1/n} f(L_n x) - f(x)| \le \epsilon + \frac{4}{\delta} \|f\|_0 \sqrt{\mathrm{Tr}\ Q_{\frac{1}{n}}}.$$

Letting $n \to \infty$ in the previous formula, we get $\lim_{n\to\infty} \sup_{x\in K} |\tilde{U}_{1/n} f(L_n x) - f(x)| = 0$ and assertion (ii) is proved. The proof is complete. ■

Lemma 7.4.10 *Let $F \in \mathcal{C}_\pi([0,T]; \mathcal{C}_b(H))$ and suppose that F is continuous on $[0,T] \times H$. Set $L_n = nR(n, A)$, $n \ge 1$. Then one has*

$$(\tilde{U}_{\frac{1}{n}} F) \circ L_n \xrightarrow{\mathcal{K}_T} F, \quad \text{as } n \to \infty,$$

where $(\tilde{U}_{\frac{1}{n}} F) \circ L_n\ (t, x) = (\tilde{U}_{\frac{1}{n}} F(t, \cdot))\,(L_n x)$, $x \in H$, $n \ge 1$, $t \in [0,T]$.

Proof We set $F_n = (\tilde{U}_{\frac{1}{n}} F) \circ L_n$, $n \ge 1$.

We argue by contradiction as for formula (7.3.57) in the proof of Theorem 7.3.5. If the thesis is not true there exist $\epsilon_0 > 0$, a compact set K of H and a sequence $t_m \subset [0,T]$ such that:

$$\sup_{x\in K} |F_{n_m}(t_m, x) - F(t_m, x)| > \epsilon_0, \quad m \ge 1. \tag{7.4.47}$$

There exists a subsequence (t_j) of (t_m) such that $t_j \to r \in [0,T]$ as $j \to \infty$. Thus we can write, setting $n_{m_j} = j$ for convenience,

$$0 < \epsilon_0 < \sup_{x\in K} |F_j(t_j,x) - F(t_j,x)| \le \Gamma_j^1 + \Gamma_j^2 + \Gamma_j^3, \quad j \ge 1$$

$$\text{where} \quad \Gamma_j^1 = \sup_{x\in K} |F_j(t_j,x) - F_j(r,x)|, \tag{7.4.48}$$

$$\Gamma_j^2 = \sup_{x\in K} |F_j(r,x) - F(r,x)|, \quad \Gamma_j^3 = \sup_{x\in K} |F(r,x) - F(t_j,x)|.$$

Now we will obtain a contradiction by showing that $\lim_{j\to\infty} \Gamma_j^1 + \Gamma_j^2 + \Gamma_j^3 = 0$.

First remark that since F is uniformly continuous on $[0,T] \times K$, it follows that $\lim_{j\to\infty} \Gamma_j^3 = 0$. Concerning Γ_j^2,

$$\Gamma_j^2 = \sup_{x\in K} |(\tilde U_{\frac{1}{j}} F(r,\cdot))\,(L_j x) - F(r,x)|$$

that goes to 0 as $n \to \infty$, by Lemma 7.4.9. It remains to consider Γ_j^1.

$$\begin{aligned}\Gamma_j^1 &= \sup_{x\in K} |(\tilde U_{\frac{1}{j}} F(t_j,\cdot))\,(L_j x) - (\tilde U_{\frac{1}{j}} F(r,\cdot))\,(L_j x)|\\ &\le \sup_{x\in K} \int_H |F(t_j,y) - F(r,y)|\,\mathcal{N}(e^{\frac{1}{j}\tilde B} L_j x, Q_{1/j})dy.\end{aligned}$$

Once we have proved that the family of measures $\mu_{j,x} = \mathcal{N}(e^{\frac{1}{j}\tilde B} L_j x, Q_{1/j})$, $j \ge 1$, $x \in K$ is tight, we obtain that $\lim_{j\to\infty} \Gamma_j^1 = 0$ and the desired contradiction follows.

Indeed suppose that $\{\mu_{j,x}\}$ is tight. Then for any $\epsilon > 0$ there exists a compact set C_ϵ of H, such that $\mu_{j,x}(H\backslash C_\epsilon) < \epsilon$ for any $j \ge 1$, $x \in K$. Using this fact we have for any $j \ge 1$

$$\Gamma_j^1 \le \sup_{x\in K} \int_{C_\epsilon} |F(t_j,y) - F(r,y)|\,\mu_{j,x}(dy) + 2\epsilon\|F\|_0.$$

Taking into account that F is uniformly continuous on $[0,T] \times C_\epsilon$, we obtain that, for j sufficiently large, $\Gamma_j^1 \le \epsilon\,(1 + 2\|F\|_0)$ and the statement is proved.

Let us verify that $\{\mu_{j,x}\}_{j\ge 1, x\in K}$ is tight. By the Prokhorov Theorem it suffices to prove that the family of measures is weakly relatively compact. To this purpose we consider any sequence (x_m) in K. There exists a subsequence (x_j) such that $x_j \to z \in K$. We prove that μ_{j,x_j} converges weakly to δ_z as $j \to \infty$. For $g \in \mathcal{C}_b(H)$, $j \ge 1$ one has:

$$\begin{aligned}&|\int_H g(y)\mathcal{N}(e^{\frac{1}{j}\tilde B} L_j x_j, Q_{1/j}) - g(z)| \le |\tilde U_{\frac{1}{j}} g(L_j x_j) - g(x_j)| + |g(x_j) - g(z)|\\ &\le \sup_{x\in K} |\tilde U_{\frac{1}{j}} g(L_j x) - g(x)| + |g(x_j) - g(z)|.\end{aligned} \tag{7.4.49}$$

Now letting $j \to \infty$, the last term of (7.4.49) tends to 0, by (ii) of Lemma 7.4.9. Therefore $\mu_{j,x}$, $j \ge 1$, $x \in K$ is tight. This completes the proof. ■

We point out that by Lemma 7.4.8 and Lemma 7.4.9 we can deduce the following result that was proved in Cerrai and Gozzi [15, §5.7] (see also Proposition 7.3.10). We provide a different and self-contained proof.

Proposition 7.4.11 *Let $\mathcal{U}$ be the generator of the Ornstein -Uhlenbeck semigroup U_t. $\mathcal{U}$ is the $\mathcal{K}$-closure of $\mathcal{U}_0$, that is: for any $f \in D(\mathcal{U})$ there exists a sequence $(f_n) \subset D(\mathcal{U}_0)$ such that*

$$f_n \xrightarrow{\mathcal{K}} f, \quad \mathcal{U}_0 f_n \xrightarrow{\mathcal{K}} \mathcal{U} f \text{ as } n \to \infty. \tag{7.4.50}$$

Proof We argue as in the proof of Proposition 7.3.46 with π-convergence replaced by $\mathcal{K}$-convergence. Take any $f \in D(\mathcal{U})$ and fix $\lambda > 0$. We set $g = (\lambda - \mathcal{U}) f \in \mathcal{C}_b(H)$. Set $g_n = (\tilde{U}_{\frac{1}{n}} g) \circ L_n$, $n \geq 1$ (with $L_n = nR(n, A)$). By Lemma 7.4.9, $g_n \in \tilde{\mathcal{C}}^2_A(H)$ for any $n \geq 1$ and moreover $g_n \xrightarrow{\mathcal{K}} g$ as $n \to \infty$.
Define $f_n = R(\lambda, \mathcal{U}) g_n$, $n \geq 1$. By Proposition 7.4.8, we know that $f_n \in D(\mathcal{U}_0)$ for any $n \geq 1$. Let us notice that since

$$\lambda f_n - \mathcal{U}_0 f_n = g_n, \quad n \geq 1,$$

if we prove that $f_n \xrightarrow{\mathcal{K}} f$ as $n \to \infty$ we deduce also that $\mathcal{U}_0 f_n \xrightarrow{\mathcal{K}} \mathcal{U} f$. Now we have

$$f_n(x) = \int_0^\infty e^{-\lambda u}\, U_u g_n(x)\, du, \quad x \in H, \ n \geq 1.$$

By a well known property of U_t (see Cerrai [14] or argue as in claim 1 of Theorem 7.3.5 with P_t replaced by U_t), for any $t \geq 0$, $U_t g_n \xrightarrow{\mathcal{K}} U_t g$ as $n \to \infty$. Taking into account that $\|U_t g_n\|_0 \leq \|g\|_0$ for $t \geq 0$ and $n \geq 1$ we can apply the Dominated Convergence Theorem in order to obtain that $f_n \xrightarrow{\mathcal{K}} f$. This completes the proof. ∎

Proof of Theorem 7.4.7. We consider the following approximations, for any $n \geq 1$, $t \in [0, T]$, $x \in H$,

$$u_n(t, x) = nR(n) U_t\, [\tilde{U}_{\frac{1}{n}} f \circ L_n](x) + \int_0^t U_{t-s} nR(n)\, [\tilde{U}_{\frac{1}{n}} F \circ L_n](s, x)\, ds, \tag{7.4.51}$$

where $R(n) = R(n, \mathcal{U})$, $L_n = nR(n, A)$, $[\tilde{U}_{\frac{1}{n}} F \circ L_n](s, x) = [\tilde{U}_{\frac{1}{n}} F(s, \cdot)](L_n x)$, $n \geq 1$, $x \in H$, $s \in [0, T]$. We claim that the maps (u_n) satisfy our assertions.

Note that the same proofs of Theorem 7.3.4 and Theorem 7.3.5 work out for this case, with $\tilde{U}_{\frac{1}{n}} f$ and $\tilde{U}_{\frac{1}{n}} F$ that are replaced respectively by

$$\tilde{U}_{\frac{1}{n}} f \circ L_n \quad \text{and} \quad \tilde{U}_{\frac{1}{n}} F \circ L_n.$$

We only point out the following facts.
Claim 1. $\tilde{U}_{\frac{1}{n}} f \circ L_n \in \tilde{\mathcal{C}}^2_A(H)$ for $n \geq 1$, by Lemma 7.4.9.
Claim 2. Applying Lemma 7.4.8, it is straightforward to obtain the following estimate, for suitable constants c_n, d_n, $n \geq 1$, $s \in [0, t]$,

$$\|U_{t-s}\tilde{U}_{\frac{1}{n}}F\circ L_n(s,\cdot)\|_{\tilde{2},A} \;\le\; \|\tilde{U}_{\frac{1}{n}}F\circ L_n(s,\cdot)\|_{\tilde{2},A}$$

$$\le\; \|L_n\|^2_{\mathcal{L}(H)}\,\|AL_n\|^2_{\mathcal{L}(H)}\,c_n\,\|\tilde{U}_{\frac{1}{n}}F(s,\cdot)\|_{\tilde{2}} \le\; c_nd_n\,\|F(s,\cdot)\|_0 \;\le\; c_nd_n\,\|F\|_0.$$

From this estimate, applying Lemma 7.3.8 (see also (7.4.38)), we get

$$\int_0^t U_{t-s}[\tilde{U}_{\frac{1}{n}}F\circ L_n](s,\cdot)\,ds \;\in\; \tilde{\mathcal{C}}^2_A(H),\;\; t\ge 0,\; n\ge 1.$$

Now by Lemma 7.4.8 and by (7.3.45), we obtain for any $t\ge 0$, $n\ge 1$,

$$u_n(t,\cdot) = nR(n)U_t[\tilde{U}_{\frac{1}{n}}f\circ L_n](\cdot) \,+\, nR(n)\int_0^t U_{t-s}[\tilde{U}_{\frac{1}{n}}F\circ L_n](s,\cdot)ds \,\in D(\mathcal{U}_0),$$

and assertion (a) is proved.

Claim 3. By Lemma 7.4.9 (assertion (ii)) we know that $\tilde{U}_{\frac{1}{n}}f\circ L_n \xrightarrow{\mathcal{K}} f$ as $n\to\infty$ and, repeating the proof of assertion (i) in Theorem 7.3.4, this is enough to prove assertion (b).

Finally to establish assertion (b'), we can adapt the proof of (i) in Theorem 7.3.5, taking into account Proposition (6.3.3) and using the following statement (see Lemma 7.4.10): $\tilde{U}_{\frac{1}{n}}F\circ L_n \xrightarrow{\mathcal{K}_T} F$ as $n\to\infty$. ■

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